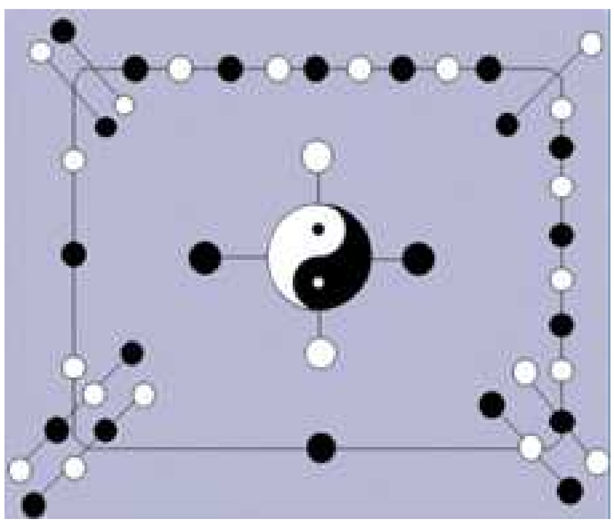
ISBN 978-1-59973-115-5

VOLUME 4, 2009

MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



January, 2010

Vol.4, 2009 ISBN 978-1-59973-115-5

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Linfan Mao
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Experience without learning is better than learning without experience.

By B.Russell, a British philosopher and mathematician.

Math.Combin.Book.Ser. Vol.4 (2009), 01-08

On the Bicoset of a Bivector Space

Agboola A.A.A.[†] and Akinola L.S.[‡]

† Department of Mathematics, University of Agriculture, Abeokuta, Nigeria

‡ Department of Mathematics and computer Science, Fountain University, Osogbo, Nigeria

E-mail: aaaola2003@yahoo.com, akinolalsb@yahoo.com

Abstract: The study of bivector spaces was first intiated by Vasantha Kandasamy in [1]. The objective of this paper is to present the concept of bicoset of a bivector space and obtain some of its elementary properties.

Key Words: bigroup, bivector space, bicoset, bisum, direct bisum, inner biproduct space, biprojection.

AMS(2000): 20Kxx, 20L05.

§1. Introduction and Preliminaries

The study of bialgebraic structures is a new development in the field of abstract algebra. Some of the bialgebraic structures already developed and studied and now available in several literature include: bigroups, bisemi-groups, biloops, bigroupoids, birings, binear-rings, bisemi-rings, biseminear-rings, bivector spaces and a host of others. Since the concept of bialgebraic structure is pivoted on the union of two non-empty subsets of a given algebraic structure for example a group, the usual problem arising from the union of two substructures of such an algebraic structure which generally do not form any algebraic structure has been resolved. With this new concept, several interesting algebraic properties could be obtained which are not present in the parent algebraic structure. In [1], Vasantha Kandasamy initiated the study of bivector spaces. Further studies on bivector spaces were presented by Vasantha Kandasamy and others in [2], [4] and [5]. In the present work however, we look at the bicoset of a bivector space and obtain some of its elementary properties.

Definition 1.1([2]) *A set $(G, +, \cdot)$ with two binary operations $+$ and $\cdot$ is called a bigroup if there exists two proper subsets G_1 and G_2 of G such that:*

(*i*) $G = G_1 \cup G_2$;

(*ii*) $(G_1, +)$ *is a group;*

[1]Received Sep.16, 2009. Accepted Oct. 8, 2009.

(iii) $(G_2, .)$ *is a group.*

Definition 1.2([2]) *A nonempty subset* H *of a bigroup* $(G, +, \cdot)$ *is called a subbigroup if* H *is itself a bigroup under* $+$ *and* $\cdot$ *defined on* G.

Theorem 1.3([2]) *Let* $(G, +, \cdot)$ *be a bigroup. The nonempty subset* H *of* G *is a subbigroup if and only if there exists two proper subsets* G_1 *and* G_2 *such that:*

(i) $G = G_1 \cup G_2$, *where* $(G_1, +)$ *and* $(G_2, .)$ *are groups;*

(ii) $(H \cap G_1, +)$ *is a subgroup of* $(G_1, +)$*;*

(iii) $(H \cap G_2, .)$ *is a subgroup of* $(G_2, .)$.

Definition 1.4([2]) *Let* $(G, +, \cdot)$ *be a bigroup where* $G = G_1 \cup G_2$. G *is said to be commutative if both* $(G_1, +)$ *and* $(G_2, \cdot)$ *are commutative.*

Definition 1.5([1]) *Let* $V = V_1 \cup V_2$ *where* V_1 *and* V_2 *are proper subsets of* V. V *is said to be a bivector space over the field* F *if* V_1 *and* V_2 *are vector spaces over the same field* F*. In this case,* V *is a bigroup.*

Definition 1.6([1]) *Let* $V = V_1 \cup V_2$ *be a bivector space. If* $\dim V_1 = m$ *and* $\dim V_2 = n$*, then* $\dim V = m + n$. *Thus there exists only* $m + n$ *linearly independent elements that can span* V*. In this case,* V *is said to be finite dimensional.*

If one of V_1 *or* V_2 *is infinite dimensional, we call* V *an infinite dimensional bivector space.*

Theorem 1.7([1]) *The bivector spaces of the same dimension over the same field need not be isomorphic in general.*

Theorem 1.8([1]) *Let* $V = V_1 \cup V_2$ *and* $W = W_1 \cup W_2$ *be two bivector spaces of the same dimension over the same field* F*. Then* V *is isomorphic to* W *if and only if* V_1 *is isomorphic to* W_1 *and* V_2 *is isomorphic to* W_2.

Example 1.9 Let $V = V_1 \cup V_2$ and $W = W_1 \cup W_2$ be two bivector spaces over a field $F = \mathcal{R}$. Suppose that $V_1 = F^4$, $V_2 = \left\{ \begin{bmatrix} v_2^1 & v_2^2 \\ 0 & v_2^3 \end{bmatrix} : v_2^i \in F, i = 1, 2, 3 \right\}$, $W_1 = P_3(F)$ (a space of polynomials of degrees ≤ 3 with coefficients in F) and $W_2 = F^3$. Clearly $\mathrm{dimV} = \mathrm{dimW} = 7$, $\dim V_1 = \dim W_1 = 4$ and $\dim V_2 = \dim W_2 = 3$. Since $V_1 \cong W_1$ and $V_2 \cong W_2$ in this case, it follows that V and W are isomorphic bivector spaces.

Theorem 1.10 *Let* $V = V_1 \cup V_2$ *be a bivector space over a field* F. *A nonempty subset* $W = W_1 \cup W_2$ *of* V *is a sub-bivector space of* V *if and only if* $W_1 = W \cap V_1$ *and* $W_2 = W \cap V_2$ *are subspaces of* V_1 *and* V_2 *respectively.*

Proof Suppose that $W = W_1 \cup W_2$ is a sub-bivector space of a bivector space $V = V_1 \cup V_2$ over F. It is clear that $W \cap V_1$ and $W \cap V_2$ are subspaces of V_1 and V_2 respectively over F. The required result follows immediately by taking $W_1 = W \cap V_1$ and $W_2 = W \cap V_2$.

Conversely, suppose that $V = V_1 \cup V_2$ is a bivector space over F and and $W = W_1 \cup W_2$ is

a nonempty subset of V such that $W_1 = W \cap V_1$ and $W_2 = W \cap V_2$ are subspaces of V_1 and V_2, respectively. We then have to show that W is a bivector space over F. To do this, it suffices to show that $W = (W \cap V_1) \cup (W \cap V_2)$. Obviously, $W \subseteq V_1 \cup W, W \cup V_2 \subseteq V$ and $W \subseteq W \cup V_2$. Now,

$$\begin{aligned}(W \cap V_1) \cup (W \cap V_2) &= [(W \cap V_1) \cup W] \cap [(W \cap V_1) \cup V_2] \\ &= [(W \cup W) \cap (V_1 \cup W)] \cap [(W \cup V_2) \cap (V_1 \cup V_2)] \\ &= [W \cap (V_1 \cup W)] \cap [(W \cup V_2) \cap V] \\ &= W \cap (W \cup V_2) \\ &= W.\end{aligned}$$

This shows that $W = (W \cap V_1) \cup (W \cap V_2)$ is a bivector space over F. □

§2. Main Results

Definition 2.1 *Let $V = V_1 \cup V_2$ be a bivector space over a field F and let $W = W_1 \cup W_2$ be a sub-bivector space of V. Let $v_0 \in V$ and $w \in W$ be such that $v_0 = v_0^1 \cup v_0^2$ and $w = w^1 \cup w^2$ where $v_0^i \in V_i, i = 1, 2$ and $w^i \in W_i, i = 1, 2$. Let P be a set defined by*

$$\begin{aligned}P &= \{v_0 + W : v_0 \in V\} \\ &= \left\{\left(v_0^1 \cup v_0^2\right) + \left(w^1 \cup w^2\right) : v_0^i \in V_i, i = 1, 2\right\} \\ &= \left\{\left(v_0^1 + W_1\right) \cup \left(v_0^2 + W_2\right) : v_0^i \in V_i, i = 1, 2\right\} \\ &= \left\{\left(v_0^1 + w^1\right) \cup \left(v_0^2 + w^2\right) : v_0^i \in V_i, w^i \in W_i, i = 1, 2\right\}.\end{aligned}$$

Then P is called a bicoset of V determined by W and v_0 is a fixed bivector in V.

Example 2.2 Let $W = W_1 \cup W_2$ be any sub-bivector space of a bivector space $V = V_1 \cup V_2$ over a field $F = \mathcal{R}$. Let $V_1 = F^3$ and $V_2 = P_2(F)$ (a space of polynomials of degrees ≤ 2 with coefficients in F). Let W_1 and W_2 be defined by

$$\begin{aligned}W_1 &= \{(a, b, c) : 3a + 2b + c = 0, a, b, c \in F\}, \\ W_2 &= \left\{p(x) : p(x) = a_2x^2 + a_1x + a_0, a_i \in F, i = 0, 1, 2\right\}.\end{aligned}$$

If $v = v_1 \cup v_2$ is any bivector in V, then $v_1 = \left(v_1^1, v_1^2, v_1^3\right) \in V_1$, where $v_1^i \in F, i = 1, 2, 3$ and also $v_2 = b_2x^2 + b_1x + b_0$, where $b_i \in F, i = 0, 1, 2$. Now, the bicoset of V determined by W is obtained as

$$\left[3\left(a - v_1^1\right) + 2\left(b - v_1^2\right) + \left(c - v_1^3\right)\right] \cup \left[(b_2 - a_2)\, x^2 + (b_1 - a_1)\, x^1 + (b_0 - a_0)\right].$$

Proposition 2.3 *Let S be a collection of bicosets of a bivector space $V = V_1 \cup V_2$ over a field F determined by sub-bivector space $W = W_1 \cup W_2$. Then S is not a bivector space over F.*

Proof Let $P = P_1 \cup P_2 = \big(v_1^1 + W_1\big) \cup \big(v_1^2 + W_2\big)$ and $Q = Q_1 \cup Q_2 = (v_2^1 + W_1) \cup (v_2^2 + W_2)$ be arbitrary members of S with $v_i^j \in V_i, i, j = 1, 2$. Clearly, $P_1 = v_1^1 + W_1, P_2 = v_1^2 + W_2, Q_1 = v_2^1 + W_1, Q_2 = v_2^2 + W_2$ are vector spaces over F and $P \cup Q = [P_1 \cup (P_1 \cup Q_1)] \cup [P_1 \cup (P_2 \cup Q_2)]$. Since $[P_1 \cup (P_1 \cup Q_1)]$ and $[P_1 \cup (P_2 \cup Q_2)]$ are obviously not vector spaces over F, it follows that S is not a bivector space over F. □

This is another marked difference between a vector space and a bivector space. We also note that $P \cap Q = [(P_1 \cap Q_1) \cup (P_2 \cap Q_1)] \cup [(P_1 \cap Q_2) \cup (P_2 \cap Q_2)]$ is also not a bivector space over F since it is a union of two bivector spaces and not a union of two vector spaces over F.

Proposition 2.4 *Let $W = W_1 \cup W_2$ be a sub-bivector space of a bivector space $V = V_1 \cup V_2$ and let $P = (v_0^1 + W_1) \cup (v_0^2 + W_2)$ be a bicoset of V determined by W where $v_0 = v_0^1 \cup v_0^2$ is any bivector in V. Then P is a sub-bivector space of V if and only if $v_0 \in W$.*

Proof Suppose that $v_0 = v_0^2 + W_2 \in W = W_1 \cup W_2$. It follows that $v_0^1 \in W_1$ and $v_0^2 \in W_2$ and consequently, $P = (v_0^1 + W_1) \cup (v_0^2 + W_2) = W_1 \cup W_2 = W$. Since W is a sub-bivector space of V , it follows that P is a sub-bivector space of V.

The converse is obvious. □

Proposition 2.5 *Let $W = W_1 \cup W_2$ be a sub-bivector space of a bivector space $V = V_1 \cup V_2$ and let $P = (v_0^1 + W_1) \cup (v_0^2 + W_2)$ and $Q = (v_1^1 + W_1) \cup (v_1^2 + W_2)$ be two bicosets of V determined by W where $v_0 = v_0^1 \cup v_0^2$ and $v_1 = v_1^1 \cup v_1^2$. Then $P = Q$ if and only if $v_0 - v_1 \in W$.*

Proof Suppose that $P = Q$. Then $(v_0^1 + W_1) \cup (v_0^2 + W_2) = (v_1^1 + W_1) \cup (v_1^2 + W_2)$ and this implies that $v_0^1 + W_1 = v_1^1 + W_1$ or $v_0^2 + W_2 = v_1^2 + W_2$ which also implies that $v_0^1 - v_1^1 \in W_1$ or $v_0^2 - v_1^2 \in W_2$ from which we obtain $(v_0^1 - v_1^1) \cup (v_0^2 - v_1^2) \in W_1 \cup W_2$ and thus $(v_0^1 \cup v_0^2) - (v_1^1 \cup v_1^2) \in W_1 \cup W_2$ that is $v_0 - v_1 \in W$.

The converse is obvious and the proof is complete. □

Proposition 2.6 *Let $P = (v_0^1 + W_1) \cup (v_0^2 + W_2)$ be a bicoset of $V = V_1 \cup V_2$ determined by $W = W_1 \cup W_2$ where $v_0 = v_0^1 \cup v_0^2$. If $v_1 = v_1^1 \cup v_1^2$ is any bivector in V such that $v_1 \in P$, then P can be expressed as $P = (v_1^1 + W_1) \cup (v_1^2 + W_2)$.*

Proof This result is obvious. □

Proposition 2.7 *Let $W = W_1 \cup W_2$ and $W' = W_3 \cup W_4$ be two distinct sub-bivector spaces of a bivector space $V = V_1 \cup V_2$ and let $P = (v_1^1 + W_1) \cup (v_1^2 + W_2)$ and $Q = (v_2^1 + W_3) \cup (v_2^2 + W_4)$ be two bicosets of V determined by W and W' respectively. If $v_0 = v_0^1 \cup v_0^2$ is any bivector in V such that $v_0 \in P$ and $v_0 \in Q$, then $P \cup Q$ is also a bicoset of V and $P \cup Q = v_0 + (W \cup W')$.*

Proof Suppose that $v_0 \in P$ and $v_0 \in Q$. It follows from Proposition 2.6 that $P = (v_0^1 + W_1) \cup (v_0^2 + W_2)$ and $Q = (v_0^1 + W_3) \cup (v_0^2 + W_4)$ and therefore

$$\begin{aligned} P \cup Q &= [(v_0^1 + W_1) \cup (v_0^2 + W_2)] \cup [(v_0^1 + W_3) \cup (v_0^2 + W_4)] \\ &= [(v_0^1 \cup v_0^2) + (W_1 \cup W_2)] \cup [(v_0^1 \cup v_0^2) + (W_3 \cup W_4)] \\ &= [v_0 + W] \cup [v_0 + W'] \\ &= v_0 + (W \cup W'). \end{aligned}$$

The required results follow. □

Definition 2.8([2]) *Let $V = V_1 \cup V_2$ be a bivector space over the field F. An inner biproduct on V is a bifunction $<,>=<,>_1 \cup <,>_2$ which assigns to each ordered pair of bivectors $x = x_1 \cup x_2$, $y = y_1 \cup y_2$ in V with $x_i, y_i \in V_i$ $(i = 1, 2)$ a pair of scalars $< x, y >=< x_1, y_1 >_1 \cup < x_2, y_2 >_2$ in F in such a way that $\forall x, y, z = z_1 \cup z_2 \in V$ and all scalars $\alpha = \alpha_1 \cup \alpha_2$ in F, the following conditions hold:*

(*i*) $< x + y, z >=< x_1 + y_1, z_1 >_1 \cup < x_2 + y_2, z_2 >_2$;

(*ii*) $< \alpha x, y >= \alpha < x, y >$;

(*iii*) $< y, x >= \overline{< x, y >}$;

(*iv*) $< x, x >> 0$ if $x \neq 0 \cup 0$.

$V = V_1 \cup V_2$ together with a specified inner biproduct $<,>=<,>_1 \cup <,>_2$ is called an inner biproduct space over the field F.

If V is a finite dimensional real inner biproduct space, it is called a Euclidean bispace. A complex inner biproduct space is called a unitary bispace.

Definition 2.9 *Let $V = V_1 \cup V_2$ be an inner biproduct space over a field F. If $x = x_1 \cup x_2$ and $y = y_1 \cup y_2$ in V with $x_i, y_i \in V_i$ $(i = 1, 2)$ are such that*

$$< x, y >=< x_1, y_1 >_1 \cup < x_2, y_2 >_2= 0 \cup 0,$$

we say that x is biorthogonal to y. If $< x, y >\neq 0 \cup 0$ but $< x_1, y_1 >_1= 0$ or $< x_2, y_2 >_2= 0$, then we say that x and y are semi biorthogonal.

If $B = B_1 \cup B_2$ is any set in $V = V_1 \cup V_2$ such that all pairs of distinct vectors in B_1 and all pairs of distinct vectors in B_2 are orthogonal, then we say that B is a biorthogonal set.

If $W = W_1 \cup W_2$ is any set in $V = V_1 \cup V_2$ and $\forall v \in V, w \in W$ with $v = v_1 \cup v_2, w = w_1 \cup w_2$ is such that

$$< v, w >=< v_1, w_1 >_1 \cup < v_2, w_2 >_2= 0 \cup 0,$$

then we call the set

$$W^\perp = W_1^\perp \cup W_2^\perp = \{v \in V :< v_1, w_1 >_1 \cup < v_2, w_2 >_2= 0 \cup 0, \forall w \in W\}$$

biorthogonal complement of W.

Definition 2.10 *Let $W_1 = W_1^1 \cup W_1^2$ and $W_2 = W_2^1 \cup W_2^2$ be sub-bivector spaces of a bivector space $V = V_1 \cup V_2$. The bisum of W_1 and W_2 denoted by $W_1 + W_2$ is defined by*

$$\begin{aligned} W_1 + W_2 &= \{(W_1^1 \cup W_1^2) + (W_2^1 \cup W_2^2) : W_i^j \subset V_i, i, j = 1, 2\} \\ &= \{(W_1^1 + W_2^1) \cup (W_1^2 + W_2^2), W_i^j \subset V_i, i, j = 1, 2\}. \end{aligned}$$

Definition 2.11 *Let $V = V_1 \cup V_2$ be a bivector space over a field F and let $W_1 = W_1^1 \cup W_1^2$ and $W_2 = W_2^1 \cup W_2^2$ be sub-bivector spaces of V. If $V_1 = W_1^1 \oplus W_2^1$ and $V_2 = W_1^2 \oplus W_2^2$, then we call*

$$\begin{aligned} V &= W_1 \oplus W_2 \\ &= (W_1^1 \oplus W_2^1) \cup (W_1^2 \oplus W_2^2) \end{aligned}$$

a direct bisum of W_1 and W_2 and any bivector $v = v_1 \cup v_2$ in V can be expressed uniquely as

$$v = (w_1^1 + w_2^1) \cup (w_1^2 + w_2^2),\ w_i^j \in W_i^j,\ i, j = 1, 2.$$

Proposition 2.12 *Let $W_1 = W_1^1 \cup W_1^2$ and $W_2 = W_2^1 \cup W_2^2$ be sub-bivector spaces of a bivector space $V = V_1 \cup V_2$. Then the bisum of W_1 and W_2 is also a sub-bivector space of V.*

Proof Obviously, $(W_1+W_2)\cap V_1$ and $(W_1+W_2)\cap V_2$ are subspaces of V_1 and V_2 respectively. Direct expansion of $[(W_1 + W_2) \cap V_1] \cup [(W_1 + W_2) \cap V_2]$ shows that

$$W_1 + W_2 = [(W_1 + W_2) \cap V_1] \cup [(W_1 + W_2) \cap V_2].$$

Consequently by Theorem 1.10 it follows that $W_1 + W_2$ is a sub-bivector space of V. $\square$

Proposition 2.13 *Let $W_1 = W_1^1 \cup W_1^2$ and $W_2 = W_2^1 \cup W_2^2$ be sub-bivector spaces of a bivector space $V = V_1 \cup V_2$. Then $V = W_1 \oplus W_2$ if and only if:*

(*i*) $V = W_1 + W_2$;

(*ii*) $W_1 \cap W_2 = \{0\}$.

Proof Suppose that $V = W_1 \oplus W_2$. Then any bivector $v = W_1^1 \cup W_1^2$ in V can be written uniquely as $v = (w_1^1 + w_2^1) \cup (w_1^2 + w_2^2),\ w_i^j \in W_i^j,\ i, j = 1, 2$, which is an element of $W_1 + W_2$ and therefore, $V = W_1 + W_2$. Also since $V = W_1 \oplus W_2$, it follows that $V_1 = W_1^1 \oplus W_2^1$ and $V_2 = W_1^2 \oplus W_2^2$. Now, let $v = W_1^1 \cup W_1^2 \in W_1 \cup W_2$. Then $v \in W_1$ and $v \in W_2$ and thus, $v \in V_1$ and $v \in V_2$. If $v \in V_1$, then we can write $v = W_1^1 \cup W_1^2 = W_2^1 \cup W_2^2$ from which we obtain

$$v_1 = w_1^1 + w_2^1 \equiv v_1 + 0, v_1 \in W_1^1, 0 \in W_2^1 \quad \text{and also,}$$

$$v_1 = w_1^1 + w_2^1 \equiv 0 + v_1, 0 \in W_1^1, v_1 \in W_2^1.$$

Since $V_1 = W_1^1 \oplus W_2^1$, it follows that $v_1 = 0$. By similar argument, we obtain $v_2 = 0$ and therefore, $v = 0 \cup 0$. Hence, $W_1 \cap W_2 = \{0\}$.

Conversely, suppose that $V = W_1 + W_2$ and $W_1 \cup W_2 = \{0\}$. Let $v = W_1^1 \cup W_1^2$ be an arbitrary bivector in V. Suppose we can write v in two ways as

$$v = W_1^1 \cup W_1^2 = (w_1^1 + w_2^1) \cup (w_1^2 + w_2^2) = (w_{11}^1 + w_{22}^1) \cup (w_{11}^2 + w_{22}^2), w_i^j, w_{ii}^j \in W_i^j, i, j = 1, 2.$$

Then we have $w_1^1 + w_2^1 = w_{11}^1 + w_{22}^1$, $w_1^2 + w_2^2 = w_{11}^2 + w_{22}^2$ from which we obtain $w_1^1 - w_{11}^1 = w_{22}^1 - w_2^1$, $w_1^2 - w_{11}^2 = w_{22}^2 - w_2^2$. But then $w_1^1 - w_{11}^1, w_1^2 - w_{11}^2 \in W_1$ and $w_{22}^1 - w_2^1, w_{22}^2 - w_2^2 \in W_2$ and since $W_1 \cup W_2 = \{0\}$, it follows that $w_1^1 - w_{11}^1 = 0 = w_1^2 - w_{11}^2$ and $w_{22}^1 - w_2^1 = 0 = w_{22}^2 - w_2^2$ from which we obtain $w_1^1 = w_{11}^1, w_1^2 = w_{11}^2$, $w_{22}^1 = w_2^1, w_{22}^2 = w_2^2$. This shows that $v \in V$ can be expressed uniquely as $v = (w_1^1 + w_2^1) \cup (w_1^2 + w_2^2), w_i^j \in W_i^j,\ i, j = 1, 2$ and hence $V = W_1 \oplus W_2$ and the proof is complete. □

Proposition 2.14 *Let $W_1 = W_1^1 \cup W_1^2$ and $W_2 = W_2^1 \cup W_2^2$ be two distinct sub-bivector spaces of a bivector space $V = V_1 \cup V_2$ such that $V = W_1 + W_2$. If $P = (v_1^1 + W_1^1) \cup (v_1^2 + W_1^2)$ and $Q = (v_2^1 + W_2^1) \cup (v_2^2 + W_2^2)$ are two bicosets of V determined by W_1 and W_2 respectively, then $P \cap Q$ is also a bicoset of V.*

Proof Suppose that $V = W_1 + W_2$. Let $v = x_1 \cup x_2$ and $u = y_1 \cup y_2$ be bivectors in V. Clearly, $u - v \in V$ and $u - v = (y_1 - x_1) \cup (y_2 - x_2) = (w_1^1 + w_2^1) \cup (w_1^2 + w_2^2), w_i^j \in W_i^j$ from which we obtain $y_1 - x_1 = w_1^1 + w_2^1, y_2 - x_2 = w_1^2 + w_2^2$ which implies that $y_1 - w_1^1 = x_1 + w_2^1, y_2 - w_1^2 = x_2 + w_2^2$ and thus, $(y_1 - w_1^1) \cup (y_2 - w_1^2) = (x_1 + w_2^1) \cup (x_2 + w_2^2) = v_0 = v_0^1 \cup v_0^2$. Since the LHS belongs to P and RHS belongs to Q, it follows that $v_0 \in P \cap Q$ and therefore, $P \cap Q$ is a bicoset of V that is $P \cap Q = v_0 + (W_1 \cap W_2)$. □

Definition 2.15 *Let $V = V_1 \cup V_2$ be a finite dimensional inner biproduct space and $W = W_1 \cup W_2$ a sub-bispace of V. Let $W^\perp = W_1^\perp \cup W_2^\perp$ be a biorthogonal complement of W and $P = (v_1 + W_1) \cup (v_2 + W_2)$ a bicoset of V determined by W, where $v = v_1 \cup v_2$ is a fixed bivector in V. It can be shown that*

$$V = W \oplus W^\perp = (W_1 \oplus W_1^\perp) \cup (W_2 \oplus W_2^\perp)$$

and consequently we have

$$W \cup W^\perp = (W_1 \cup W_1^\perp) \cup (W_2 \cup W_2^\perp) = \{0\} \cup \{0\}.$$

Suppose that $x = x_1 \cup x_2$ and $y = y_1 \cup y_2$ are bivectors such that $x_i \in W_i$ and $y_i \in W_i^\perp, i = 1, 2$. Suppose also that $v = v_1 \cup v_2 = (x_1 + y_1) \cup (x_2 + y_2)$. Then P can be represented by

$$\begin{aligned} P &= (x_1 + y_1 + W_1) \cup (x_2 + y_2 + W_2) \\ &= (y_1 + W_1) \cup (y_2 + W_2),\ \text{since } x_i \in W_i, i = 1, 2. \end{aligned}$$

This representation is called the biprojection of v on W and it is unique.

To establish the uniqueness, let $z = z_1 \cup z_2$ be any bivector in $W^\perp$ and let P have another representation $P = (z_1 + W_1) \cup (z_2 + W_2), z_i \in W_i^\perp, i = 1, 2$. Then we have $y_1 + W_1 = z_1 + W_1$,

$y_2 + W_2 = z_2 + W_2$ so that $y_1 - z_1 \in W_1$, $y_2 - z_2 \in W_2$ and thus $y_1 - z_1 \in W_1 \cap W_1^{\perp} = \{0\}$, $y_2 - z_2 \in W_2 \cap W_2^{\perp} = \{0\}$ which implies that $y_1 - z_1 = 0$, $y_2 - z_2 = 0$ from which we obtain $y_1 = z_1$, $y_2 = z_2$ and the uniqueness of P is established.

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Math.Combin.Book.Ser. Vol.4 (2009), 09-19

Smarandachely Bondage Number of a Graph

Karam Ebadi

Department of Studies in Mathematics of University of Mysore

Manasagangotri, Mysore-5700 06, Karnataka, India.

L.Pushpalatha

Department of Mathematics of Yuvaraja's College, Mysore, India.

E-mail: *Karam_Ebadi@yahoo.com*, *pushpakrishna@yahoo.com*

Abstract: A dominating set D of a graph G is called a *Smarandachely dominating s-set* if for an integer s, each vertex v in $V - D$ is adjacent to a vertex $u \in D$ such that $\deg u + s = \deg v$. The minimum cardinality of Smarandachely dominating s-set in a graph G is called the *Smarandachely dominating s-number* of G, denoted by $\gamma_S^s(G)$. Such a set with minimum cardinality is called a *Smarandachely dominating s-set*. The *Smarandachely bondage s-number* $b_S^s(G)$ of a graph G is defined to be the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma_S^s(G - E') > \gamma_S^s(G)$. Particularly, the set with minimum Smarandachely bondage s-number for all integers $s \geq 0$ or $s \leq 0$ is called the *strong* or *weak dominating number* of G, denoted by $\gamma_s(G)$ or $\gamma_w(G)$, respectively. In this paper, we present some bounds on $b_s(G)$ and $b_w(G)$ and give exact values for $b_s(G)$ and $b_w(G)$ for complete graphs, paths, wheels and bipartite complete graphs. Some general bounds are also given.

Key Words: Smarandachely dominating s-set, Smarandachely dominating s-number, Smarandachely bondage s-number, strong or weak bondage numbers.

AMS(2000): 05C69.

§1. Introduction

In this paper, we follow the notation of [6,7]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. A set $D \subseteq V$ is a dominating set of G if every vertex v in $V - D$ there exists a vertex u in D such that u and v are adjacent in G. The domination number of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. The concept of domination in graphs, with its many variations, is well studied in graph theory. A thorough study of domination appears in [6,7]. Let $uv \in E$. Then, u and v dominate each other. A dominating set D of a graph G is called a *Smarandachely dominating s-set* if for an integer s, each vertex

[1]Received Sep.28, 2009. Accepted Oct. 12, 2009.

v in $V-D$ is adjacent to a vertex $u \in D$ such that $\deg u + s = \deg v$. The minimum cardinality of Smarandachely dominating s-set in a graph G is called the *Smarandachely dominating s-number* of G, denoted by $\gamma_S^s(G)$. Such a set with minimum cardinality is called a *Smarandachely dominating s-set*. The *Smarandachely bondage s-number* $b_S^s(G)$ of a graph G is defined to be the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma_S^s(G-E') > \gamma_S^s(G)$. Particularly, the set with minimum Smarandachely bondage s-number for all integers $s \geq 0$ or $s \leq 0$ is called the *strong* or *weak dominating number* of G, denoted by $\gamma_s(G)$ or $\gamma_w(G)$, respectively.

As a special case of Smarandachely bondage number, the strong (weak) domination was introduced by E. Sampathkumar and L.Pushpa Latha in [8]. For any undefined term, we refer Harary [4]. By definition, the bondage number $b(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges $E' \subseteq E$ for which $\gamma(G-E') > \gamma(G)$. Thus, the bondage number of G is the smallest number of edges whose removal renders every minimum dominating set of G a nondominating set in the resulting spanning subgraph. Since the domination number of every spanning subgraph of a nonempty graph G is at least as great as $\gamma(G)$, the bondage number of a nonempty graph is well defined. This concept was introduced by Bauer, Harary, Nieminen and Suffel [1] and has been further studied by Fink, Jacobson, Kinch and Roberts [2], Hartnell and Rall [5], etc. The strong bondage number of G, denoted $b_s(G)$, as the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma_s(G-E') > \gamma_s(G)$. This concept was introduced by J. Ghoshal, R. Laskar, D. Pillone and C. Wallis [3].

We define the weak bondage number of G, denoted $b_w(G)$, as the minimum cardinality among all sets of edges $E' \subseteq E$ such that $\gamma_w(G-E') > \gamma_w(G)$, and we deal with the strong bondage number of a nonempty graph G.

§2. Exact Values for $b_s(G)$ and $b_w(G)$

We begin our investigation of the strong and weak bondage numbers by computing its value for several well known classes of graphs. In several instances we shall have cause to use the ceiling function of a number x. This is denoted $\lceil x \rceil$ and takes the value of the least integer greater than or equal to x. We begin with a rather straightforward evaluation of the strong and weak bondage numbers of the complete graph of order n.

Proposition 2.1 *The strong bondage number of the complete graph K_n $(n \geq 2)$ is*

$$b_s(K_n) = \lceil n/2 \rceil.$$

Proof. Let $u_1, u_2, ..., u_n$ be the n vertices of degree $n-1$. Then clearly removal of fewer than $n/2$ edges results in a graph H having maximum degree $n-1$. Hence $b_s(K_n) \geq \lceil n/2 \rceil$. Now we consider the following cases.

Case 1. If n is even, then the removal of $n/2$ independent edges $u_1u_2, u_3u_4, ..., u_{n-1}u_n$ results in a graph H' regular of degree $n-2$. Hence $b_s(K_n) = n/2$.

Case 2. If n is odd, then the removal of $(n-1)/2$ independent edges $u_1u_2, u_3u_4, ..., u_{n-2}u_{n-1}$

yields a graph H'' containing exactly one vertex u_n of degree $n-1$. Thus by removing an edge incident with u_n we obtain a graph H''' with maximum degree $n-2$. Hence $b_s(K_n) = (n-1)/2+1$.

Combining cases (1) and (2) it follows that $b_s(K_n) = \lceil n/2 \rceil$. □

Proposition 2.2 *The weak bondage number of the complete graph K_n $(n \geq 2)$ is*

$$b_w(K_n) = 1.$$

Proof If H is a spanning subgraph of K_n that is obtained by removing any edge from K_n, then H contains two vertices of degree $n-2$. Whence $\gamma_w(H) = 2 > 1 = \gamma_w(K_n)$. Hence $b_w(K_n) = 1$. □

If G is a regular graph, then $\gamma(G) = \gamma_s(G)$ because in a regular graph, the degrees of all the vertices are equal. We next consider paths P_n and cycles C_n on n vertices and find that $\gamma(C_n) = \gamma_s(C_n)$ because C_n is a regular graph. Also $\gamma(P_n) = \gamma_s(P_n)$ since we can choose from all the γ sets of P_n, one which dose not include either end vertex. Such a γ set is also a γ_s set and hence we get $\gamma(P_n) \geq \gamma_s(P_n)$ but since $\gamma(G) \geq \gamma_s(G)$ for all graphs G, which follows

Lemma 2.3 *The strong domination number of the n-cycle and the path of order n are respectively*

(*i*) $\gamma_s(C_n) = \lceil n/3 \rceil$ *for* $n \geq 3$ *and*

(*ii*) $\gamma_s(P_n) = \lceil n/3 \rceil$ *for* $n \geq 2$.

Lemma 2.4 *The weak domination number of the $n-$ cycle and the path of order n are respectively*

(*i*) $\gamma_w(C_n) = \lceil n/3 \rceil$ *for* $n \geq 3$ *and*

(*ii*)

$$\gamma_w(P_n) = \begin{cases} \lceil n/3 \rceil & \text{if } n \equiv 1 \text{ (mod3)}, \\ \lceil n/3 \rceil + 1 & \text{otherwise.} \end{cases}$$

Proof (*i*) Since C_n is a regular graph, so $\gamma_w(C_n) = \gamma(C_n)$ and proof techniques in [2].

(*ii*) $\gamma_w(P_n) = \lceil (n-4)/3 \rceil + 2 = \gamma(P_{n-4}) + 2$, the proof is the same as in [2]. □

Theorem 2.5 *The strong bondage number of the n-cycle (with $n \geq 3$) is*

$$b_s(C_n) = \begin{cases} 3 & \text{if } n \equiv 1 \text{ (mod3)}, \\ 2 & \text{otherwise.} \end{cases}$$

Proof Since $\gamma_s(C_n) = \gamma_s(P_n)$ for $n \geq 3$, we see that $b_s(C_n) \geq 2$. If $n \equiv 1$ (mod3) the removal of two edges from C_n leaves a graph H consisting of two paths P and Q. If P has order n_1 and Q has order n_2, then either $n_1 \equiv n_2 \equiv 2$ (mod3), or, without loss of generality,

$n_1 \equiv 0$ (mod3) and $n_2 \equiv 1$ (mod3). In the former case,

$$\begin{aligned}\gamma_s(H) &= \gamma_s(P)+\gamma_s(Q) = \lceil n_1/3\rceil + \lceil n_2/3\rceil \\ &= (n_1+1)/3+(n_2+1)/3 = (n_1+n_2+2)/3 = (n+2)/3 = \lceil n/3\rceil = \gamma_s(C_n).\end{aligned}$$

In the latter case.

$$\gamma_s(H) = \gamma_s(P)+\gamma_s(Q) = n_1/3+(n_2+2)/3 = (n+2)/3 = \lceil n/3\rceil = \gamma_s(C_n).$$

In either case, when $n \equiv 1$ (mod3) we have $b_s(C_n) \geq 3$. Now we consider two cases.

Case 1 Suppose that $n \equiv 0, 2$ (mod3). The graph H obtained removing two adjacent edges from C_n consist of an isolated vertex and a path of order $n-1$. Thus

$$\gamma_s(H) = \gamma_s(P_1)+\gamma_s(P_{n-1}) = 1+\lceil (n-1)/3\rceil = 1+\lceil n/3\rceil = 1+\gamma_s(C_n),$$

Whence $b_s(C_n) \leq 2$ in this case. Combining this with the upper strong bondage obtained earlier, we have $b_s(C_n) = 2$ if $n \equiv 0, 2$ (mod3).

Case 2 Suppose now that $n \equiv 1$ (mod3). The graph H resulting from the deletion of three consecutive edges of C_n consists of two isolated vertices and a path of order $n-2$. Thus,

$$\gamma_s(H) = 2+\lceil (n-2)/3\rceil = 2+(n-1)/3 = 2+(\lceil n/3\rceil - 1) = 1+\gamma_s(C_n),$$

So that $b_s(C_n) \leq 3$. With the earlier inequality we conclude that $b_s(C_n) = 3$ when $n \equiv 1$ (mod3). □

Theorem 2.6 *The weak bondage number of the n-cycle (with $n \geq 3$) is*

$$b_w(C_n) = \begin{cases} 2 & \text{if } n \equiv 1 \text{ (mod3)}, \\ 1 & \text{otherwise.} \end{cases}$$

Proof Assume $n \not\equiv 1$ (mod3) since $\gamma_w(P_n) = \lceil n/3\rceil + 1 = \gamma_w(C_n)+1 > \gamma_w(C_n)$. Hence $b_w(C_n) = 1$. Now assume $n \equiv 1$ (mod3) since $\gamma_w(C_n) = \gamma_w(P_n)$ it follows that $b_w(C_n) \geq 2$.

Let H be the graph obtained by the removal of two edges from C_n such that P_3 and P_{n-3} are formed. Then $\gamma_w(H) = \gamma_w(P_3)+\gamma_w(P_{n-3}) = 2+\lceil (n-3)/3\rceil = 2+\lceil n/3\rceil - 1 = \lceil n/3\rceil + 1 > \gamma_w(C_n)$. Hence $b_w(C_n) \leq 2$ thus $b_w(C_n) = 2$. □

As an immediate Corollary to Theorem 2.5 we have the following.

Corollary 2.7 *The strong bondage number of the path (with $n \geq 3$) is given by*

$$b_s(P_n) = \begin{cases} 2 & \text{if } n \equiv 1 \text{ (mod3)}, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 2.8 *The weak bondage number of the path (with $n \geq 3$) is*

$$b_w(P_n) = \begin{cases} 2 & \text{if } n = 3, 5, \\ 1 & \text{otherwise.} \end{cases}$$

Proof It is easy to verify that $b_w(P_n) = 2$ for $n = 3, 5$.

Let H be the graph obtained by the removal of one edge from P_n such that P_3 and P_{n-3} are formed. Then $\gamma_w(H) = \gamma_w(P_3) + \gamma_w(P_{n-3})$. Now we consider the following cases.

Case 1 If $n \equiv 1$ (mod3) then $\gamma_w(H) = \gamma_w(P_3) + \gamma_w(P_{n-3}) = 2 + \lceil (n-3)/3 \rceil = 2 + \lceil n/3 \rceil - 1 = \lceil n/3 \rceil + 1$ then $\gamma_w(H) > \gamma_w(P_n)$. Hence $b_w(P_n) = 1$.

Case 2 If $n \not\equiv 1$ (mod3) we have $\gamma_w(H) = 2 + \lceil (n-3)/3 \rceil + 1 = 2 + \lceil n/3 \rceil - 1 + 1 = 2 + \lceil n/3 \rceil > \gamma_w(P_n)$ then $\gamma_w(H) > \gamma_w(P_n)$. Hence $b_w(P_n) = 1$. □

Lemma 2.9 *The strong and weak domination numbers of the wheel W_n (with $n \geq 4$) are*

(*i*) $\gamma_s(W_n) = 1$;

(*ii*) $\gamma_w(W_n) = \lceil (n-1)/3 \rceil$.

Proof (*i*) Since $\gamma(W_n) = \gamma_s(W_n)$ so proof techniques same in [2].

(*ii*) Since $\gamma_w(W_n) = \gamma(C_{n-1}) = \lceil (n-1)/3 \rceil$ so proof techniques same in [2]. □

Proposition 2.10 *The strong bondage number of the wheel W_n (with $n \geq 4$) is $b_s(W_n) = 1$.*

Proof Let x be the vertex of maximum degree of W_n. Let v be a vertex of W_n such that *deg* $v <$ *deg* x. Let H be the graph obtained from W_n by removing edge xv. Then no one vertex strongly dominates H. So $\gamma_s(W_n - xv) > \gamma_s(W_n)$. Hence $b_s(W_n) = 1$. □

Proposition 2.11 *The weak bondage number of W_n (with $n \geq 4$) is given by*

$$b_w(W_n) = \begin{cases} 2 & \text{if } n \equiv 2 \text{ (mod3)}, \\ 1 & \text{otherwise.} \end{cases}$$

Proof Assume $n \equiv 0, 1$ $(mod 3)$, let e be an edge on the $(n-1)$-cycle. Then $\gamma_w(W_n - e) = \lceil (n-5)/3 \rceil + 2 = \lceil (n-2)/3 \rceil + 1 = \lceil (n-1)/3 \rceil + 1 > \lceil (n-1)/3 \rceil = \gamma_w(W_n)$, whence $b_w(W_n) = 1$.

Now assume $n \equiv 2$ $(mod 3)$, the removal of any one edge from W_n will not alter $\gamma_w(W_n)$. So when $n \equiv 2$ $(mod 3)$ we have $b_w(W_n) \geq 2$.

Let H be the graph obtained by the removal of two adjacent edges from W_n such that these edges are not incident with the vertex of maximum degree. Then $\gamma_w(H) = \lceil (n-6)/3 \rceil + 3 = \lceil n/3 \rceil + 1 = \lceil (n-1)/3 \rceil + 1 > \lceil (n-1)/3 \rceil = \gamma_w(W_n)$, whence $b_w(W_n) = 2$. □

Lemma 2.12 *The strong and weak domination numbers of the $K_{r,t}$ are*

(*i*)

$$\gamma_s(K_{r,t}) = \begin{cases} 2 & \text{if } 2 \leq r = t, \\ \text{r} & \text{if } 1 \leq r < t. \end{cases}$$

(ii)

$$\gamma_w(K_{r,t}) = \begin{cases} \text{t} & \text{if } 1 \leq r < t, \\ 2 & \text{if } 2 \leq r = t. \end{cases}$$

Proof (i) see [3].

(ii) Note that the vertices in the second partite set have the smallest degree. If $1 \leq r < t$, then to weakly dominate these vertices, we need include all of them in any wd-set and these suffice to weakly dominate the rest. If $r = t \geq 2$, we claim $\gamma_w = 2$. Since $t \geq 2$, none of the vertices in the graph are of full degree hence γ_w in this case is greater than 1. Now to demonstrate a wd-set of cardinality 2, we can take one vertex from the first partite set which weakly dominate the rest of the vertices in the first partite set, we use a vertex from the second partite set. Note that a vertex from the second partite set has equal degree as the vertices in the first set since $r = t$. □

The next theorem establishes the strong and weak bondage numbers of the complete bipartite graph $K_{r,t}$.

Theorem 2.13 *Let $K_{r,t}$ be a complete bipartite graph, where $4 \leq r \leq t$, then*

$$b_s(K_{r,t}) = \begin{cases} \text{2r} & \text{if } t = r + 1, \\ \text{r} & \text{otherwise.} \end{cases}$$

Proof Let $V = V_1 \cup V_2$ be the vertex set of $K_{r,t}$ such that $|V_1| = r$ and $|V_2| = t$. We consider the following cases.

Case 1 Suppose $t = r + 1$ and $v \in V_2$, then by removing all edges incident whit v, we obtain a graph H containing two components K_1 and $K_{r,t-1}$. Hence
$\gamma_s(H) = \gamma_s(K_1) + \gamma_s(K_{r,t-1}) = 1 + 2 < r = \gamma_s(K_{r,t})$. Now let $v \in V_2$ and $u \in V_1$ be a vertex of $K_{r,t}$, then by removing all edges incident to both u and v, we obtain a graph H containing two components $2K_1$ and $K_{r-1,t-1}$, thus

$$\gamma_s(H) = 2\gamma_s(K_1) + \gamma_s(K_{r-1,t-1}) = 2 + r - 1 = r + 1 > r = \gamma_s(K_{r,t}).$$

Hence

$$b_s(K_{r,t}) = deg\ u + deg\ v - 1 = |V_2| + |V_1| - 1 = t + r - 1 = 2r$$

for $t = r + 1$.

Case 2 Suppose $r = t$, then by Lemma 2.12, $\gamma_s(K_{r,t}) = 2$. Let $v \in V_2$, then by removing all edges incident whit v, we obtain a graph H containing two components K_1 and $K_{r,t-1}$, thus
$\gamma_s(H) = \gamma_s(K_1) + \gamma_s(K_{r,t-1}) = 1 + t - 1 = t = r > 2 = \gamma_s(K_{r,t})$. Hence $b_s(K_{r,t}) = deg\ v = |V_1| = r$ for $r = t$.

Case 3 Suppose $r+1 < t$, then by Lemma 2.12, $\gamma_s(K_{r,t}) = r$. Let $v \in V_2$, then by removing all edges incident whit v, we obtain a graph H containing two components K_1 and $K_{r,t-1}$. Hence

$\gamma_s(H) = \gamma_s(K_1) + \gamma_s(K_{r,t-1}) = 1 + r > r = \gamma_s(K_{r,t})$. Thus $b_s(K_{r,t}) = deg\ v = |V_1| = r$ for $r + 1 < t$. □

Theorem 2.14 *Let $K_{r,t}$ be a complete bipartite graph, where $1 \leq r \leq t$, then $b_w(K_{r,t}) = t$.*

Proof Let $V = V_1 \cup V_2$ be the vertex set of $K_{r,t}$ where $|V_1| = r$ and $|V_2| = t$. Let $v \in V_1$ and $r = t \geq 2$, then by removing all edges incident whit v, we obtain a graph H containing two components K_1 and $K_{r-1,t}$. Hence

$\gamma_w(H) = \gamma_w(K_1) + \gamma_w(K_{r-1,t}) = 1 + t > 2 = \gamma_w(K_{r,t})$. Thus

$b_w(K_{r,t}) = deg\ v = |V_2| = t$.

Now suppose $r < t$ and $v \in V_1$, then by removing all edges incident whit v, we obtain a graph H containing two components K_1 and $K_{r-1,t}$. Hence

$\gamma_w(H) = \gamma_w(K_1) + \gamma_w(K_{r-1,t}) = 1 + t > t = \gamma_w(K_{r,t})$. Thus

$$b_w(K_{r,t}) = \text{deg} v = |V_2| = t.$$ □

§3. The Strong and Weak Bondage Numbers of a Tree

We now consider the strong and weak bondage numbers for a tree T. Define a support to be a vertex in a tree which is adjacent to an end-vertex (see [3]).

Proposition 3.1 *Every tree T with $(n \geq 4)$ has at least one of the following characteristics.*

(1) A support adjacent to at least 2 end-vertex.

(2) A support is adjacent to a support of degree 2.

(3) A vertex is adjacent to 2 support of degree 2.

(4) The support of a leaf and the vertex adjacent to the support are both of degree 2.

Proof See [3] for the proof. □

Theorem 3.2 *If T is a nontrivial tree then $b_s(T) \leq 3$.*

Proof See [3] for the proof. □

Proposition 3.3 *If any vertex of tree T is adjacent with two or more end-vertices, then $b_s(T) = 1$.*

Proof Let u be a cut vertex adjacent two or more end-vertices. Then u belongs to every minimum strong dominating set of T. Let v be an end-vertex adjacent to u. Then $T - uv$ contains an isolated vertex and a tree T' of order $n - 1$. Therefore $\gamma_s(T - uv) = \gamma_s(T') + 1 > \gamma_s(T)$. Hence $b_s(T) = 1$. □

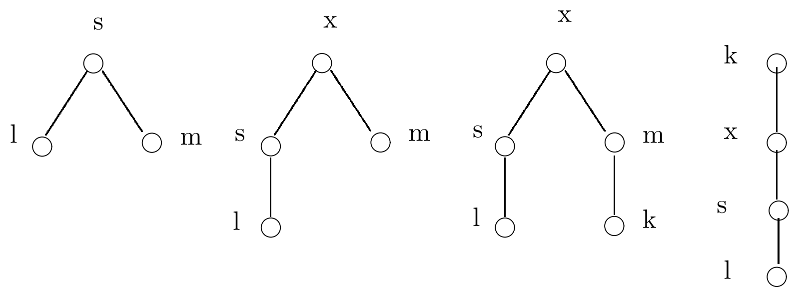

Fig.1: End characteristics of trees in Case 2 of the Proof of Theorem 3.4

Theorem 3.4 *If T is a nontrivial tree, then $b_w(T) \leq \Delta(T)$.*

Proof The statement is obviously true for trees order 2 or 3, so we shall suppose that T has at least 4 vertices. Now we consider the following cases.

Case 1 Suppose T has a support vertex s that is adjacent to two (and possibly more) end-vertex, that dose not belong to a weak dominating set. Let E_s denote the set of edges incident with s. And let D be a minimum weak dominating set for $T - E_s$. Then s is in D and $D \setminus \{s\}$ is a weak dominating set for T. Hence $\gamma_w(T - E_s) > \gamma_w(T)$ thus $b_w(T) \leq |E_s| = deg\ s \leq \Delta(T)$.

Case 2 Suppose a support vertex is adjacent to a support vertex of degree 2. Delete the edge (s, l). The vertex x then has two end-vertices an adjacent to s and m. Let D be wd-set of $T - \{(s, l)\}$. Then s is in D and $D \setminus \{s\}$ is a weak dominating set for T. Hence $b_w(T)$ in this case equals 1.

Case 3 In this case delete the edge (s, l). If $\gamma_w(T - \{(s, l)\}) < \gamma_w(T)$, then it will contradict the assumption that the γ_w-set was the smallest wd-set for T. If $\gamma_w(T - \{(s, l)\})$ is greater that $\gamma_w(T)$ then we have done. If $\gamma_w(T - \{(s, l)\}) = \gamma_w(T)$, then the vertex x has a one support vertex s in $T - \{(s, l)\}$, that adjacent to it. then by Case 2, deleting on more edge $(\{m, k\})$ will increase the weak domination number of the resulting graph. So in this case $b_w(T) = 2$.

Case 4 In the last case, either s or l is any weak dominating set of T. By removing edges (k, x) and (x, s), we make the necessary for any γ_w-set for the resulting graph to contain x and so $b_w(T) = 2$ this completes the proof. □

Theorem 3.5 *Let T be a tree. Then $b_w(T) = \Delta(T)$ if and only if $T = K_{1,r}$.*

Proof This follows from Theorem 3.4. □

§4. General Bounds on Strong and Weak Bondage Numbers

Proposition 4.1([2]) *If G is a nonempty graph, then*

$$b(G) \leq min\{deg\ u + deg\ v - 1 : \ u \ and \ v \ are \ adjacent\}.$$

Theorem 4.2 *If $\gamma(G) = \gamma_s(G)$ and $\gamma(G) = \gamma_w(G)$ then,*

(*i*) $b_s(G) \leq b(G)$;

(*ii*) $b_w(G) \leq b(G)$.

Proof Let E be a b-set of G. Then $\gamma_s(G) = \gamma(G) < \gamma(G - E) \leq \gamma_s(G - E)$. Thus $b_s(G) \leq b(G)$ and for (ii) proof is same. □

Theorem 4.3 *If G is a nonempty graph and $\gamma(G) = \gamma_s(G)$ then*

$$b_s(G) \leq min\{deg\, u + deg\, v - 1 \mid \; u \;\; and \;\; v \;\; are \;\; adjacent\}.$$

Proof This follows from Proposition 4.1 and Theorem 4.2. □

Theorem 4.4 *For any graph G,*

$$b_s(G) \leq q - p + \gamma_s(G) + 1$$

Proof Let D be a γ_s-set of a graph G. For each vertex $v \in V \setminus D$ choose exactly one edge which is incident to v and to a vertex in D. Let E_0 be the set of all such edges. Then clearly $\gamma_s(G - (E - E_0)) = \gamma_s(G)$ and $|E - E_0| = q - p + \gamma_s(G)$. So for any edge $e \in G - (E - E_0) = E_0$ we see that $\{E - E_0\} \cup \{e\}$ is a strong bondage set of G. Thus

$$b_s(G) \leq q - p + \gamma_s(G) + 1 \quad \square$$

Corollary 4.5 *For any graph G,*

$$b_s(G) \leq q - \Delta(G) + 1$$

Proof In [8], We have known that $\gamma_s(G) \leq p - \Delta(G)$. By applying Theorem 4.4, we get that $b_s(G) \leq q - \Delta(G) + 1$. □

Theorem 4.6 *If G is a nonempty graph with strong domination number $\gamma_s(G) \geq 2$, Then*

$$b_s(G) \leq (\gamma_s(G) - 1)\Delta(G) + 1.$$

Proof We proceed by induction on the strong domination number $\gamma_s(G)$. Let G be a nonempty graph with $\gamma_s(G) = 2$, and assume that $b_s(G) \geq \Delta(G) + 2$, then, if u is a vertex of maximum degree in G, we have $\gamma_s(G - u) = \gamma_s(G) - 1 = 1$, and $b_s(G - u) \geq 2$. Since $\gamma_s(G) = 2$ and $\gamma_s(G - u) = 1$, there is a vertex v that is adjacent with every vertex of G but u, that $deg_G v = \Delta(G)$ also, and u is adjacent with every vertex of G except v. Since $b_s(G - u) \geq 2$, the removal from $G - u$ of any one edge incident with v again leaves a graph with strong domination number 1. Thus there is a vertex $w \neq v$ that is adjacent with every vertex of $G - u$. But, since v is the only vertex of G that is not adjacent with u, vertex w must be adjacent in G with u. This however implies that $\gamma_s(G) = 1$, a contradiction. Thus $b_s(G) \leq \Delta(G) + 1$ if $\gamma_s(G) = 2$.

Now, let $(k \geq 2)$ be any integer for which the following statement is true: If H is nonempty graph with $\gamma_s(G) = k$, then $\gamma_s(H) \leq (k-1) \cdot \Delta(H) + 1$. Let G be a graph nonempty graph with

$\gamma_s(G) = k+1$, and assume that $b_s(G) > k\cdot\Delta(G)+1$. Then. But then, $b_s(G) \leq b_s(G-u)+deg\ u$, and by the inductive hypothesis we have

$$b_s(G) \leq [(k-1)\cdot\Delta(G-u)+1] + deg\ u \leq (k-1)\cdot\Delta(G)+1+\Delta(G),$$

or

$$b_s(G) \leq k\cdot\Delta(G)+1,$$

a contradiction to our assumption that $b_s(G) > k\cdot\Delta(G)+1$. Thus, $b_s(G) \leq k\cdot\Delta(G)+1$, and, by the principle of mathematical induction, the proof is complete. □

Theorem 4.7 *If G is a planar graph, then*

$$b_w(G) \leq \Delta(G).$$

Proof Suppose G has a vertex u with maximum degree that dose not belong to a weak dominating set. Let E_u denote the set of edges incident with u. And let D be a minimum weak dominating set for $G - E_u$. Then u is in D and $D \setminus u$ is a weak dominating set for G. Hence $\gamma_w(G-E_u) > \gamma_w(G)$ thus $b_w(G) \leq |E_u| = deg\ u \leq \Delta(G)$. □

§5. Open Problems

We strongly believe the following to be true.

Theorem 5.1 *If G is a nonempty graph of order $(n \geq 2)$ then $b_w(G) \leq n-1$.*

Theorem 5.2 *If G is a nonempty graph of order $(n \geq 2)$ then $b_w(G) \leq n-\delta(G)$.*

Theorem 5.3 *If G is a nonempty graph of order $(n \geq 2)$ then $b_s(G) \leq n-1$.*

Other bounds for the strong and weak bondage of a graph exist. For several classes of graphs, $b_s(G) \leq \Delta(G)$ and $b_w(G) \leq \Delta(G)$. Let F be the set of edges incident with a vertex of maximum degree. Then it can be shown that $\gamma_s(G-F) \geq \gamma_s(G)$ and similarly $\gamma_w(G-F) \geq \gamma_w(G)$. But it is not necessary that this action would result in an increase in the strong and weak domination numbers. See Fig.2. The calculation for the strong and weak bondage for multipartite graphs remains open. Unions, joins and product of graphs could be investigated for their strong and weak bondage in terms of the constituent graphs. This implies that we need to calculate the strong and weak domination of these graphs. The problem of strong and weak domination is virtually unexplored and so there are several classes of graphs for which the strong and weak domination numbers could be calculated.

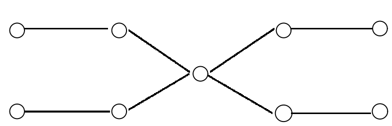

Fig. 2

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Math.Combin.Book.Ser. Vol.4 (2009), 20-30

Domination Number in 4-Regular Graphs

H.Abdollahzadeh Ahangar

(Department of Basic Science, Babol University of Technology, Babol, Iran)

Pushpalatha L.

(Departmant of Mathematics, Yuvaraja's College, Mysore-570005, India)

E-mail: ha.ahangar@yahoo.com, pushpakrishna@yahoo.com

Abstract: A set of vertices S in a graph G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S. Particularly, if $k = 1$, such a set is called a dominating set of G. The *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. For abbreviation, we denote $\gamma_1(G)$ by $\gamma(G)$. In [9], Reed proved that the domination number $\gamma(G)$ of every $n-$vertex graph G with minimum degree at least 3 is at most $3n/8$. In this note, we present a sequence of Hamiltonian 4-regular graphs whose domination numbers are sharp. Here we state some results which will pave the way in characterization of domination number in regular graphs. Also, we determine independent, connected, total and forcing domination number of those graphs.

Key Words: Regular graph, Smarandachely k-dominating set, Hamiltonian graph.

AMS(2000): 05C69.

§1. Introduction

Throughout this paper, all graphs considered are finite, undirected, loopless and without multiple edges. We refer the reader to [11] for terminology in graph theory.

Let $G = (V, E)$ be a graph with vertex set V and edge set E, and let $v \in V$. The neighborhood of v, denoted by $N(v)$, is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u \in V | uv \in E\}$. For $S \subseteq V$, the neighborhood of S, denoted by $N(S)$, is defined by $N(S) = \cup_{v \in S} N(v)$, and the closed neighborhood $N[S]$ of S is the set $N[S] = N(S) \cup S$ and the degree of x is $deg_G(x) = |N_G(x)|$.

A set of vertices S in a graph G is said to be a *Smarandachely k-dominating set*, if each vertex of G is dominated by at least k vertices of S. Particularly, if $k = 1$, such a set is called a *dominating set* of G. The *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum

[1]Received Sep.28, 2009. Accepted Oct. 14, 2009.

cardinality of a Smarandachely k-dominating set of G. For abbreviation, we denote $\gamma_1(G)$ by $\gamma(G)$. The domination number has received considerable attention in the literature.

A dominating set S is called a *connected dominating set* if the subgraph $G[S]$ induced by S is connected. The *connected domination number* of G denoted by $\gamma_c(G)$ is the minimum cardinality of a connected dominating set of G. A dominating set S is called an *independent dominating set* if S is an independent set. The *independent domination number* of G denoted by $i(G)$ is the minimum cardinality of an independent dominating set of G. A dominating set S is a *total dominating set* of G if $G[S]$ has no isolated vertex and the *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A subset F of a minimum dominating set S is a *forcing subset* for S if S is the unique minimum dominating set containing F. The *forcing domination number* $f(G,\gamma)$ of S is the minimum cardinality among the forcing subsets of S, and the forcing domination number $f(G,\gamma)$ of G is the minimum forcing domination number among the minimum dominating sets of G ([1]-[7]). For every graph G, $f(G,\gamma) \leq \gamma(G)$.

The problem of finding the domination number of a graph is NP-hard, even when restricted to 4-regular graphs. One simple heuristic is the greedy algorithm [10]. Let d_g be the size of the dominating set returned by the greedy algorithm. In 1991 Parekh [8] showed that $d_g \leq n+1-\sqrt{2e+1}$. Reed [9] proved that $\gamma(G) \leq \frac{3}{8}n$. Fisher et al. [3]-[4] repeated this result and showed that if G has girth at least 5 then $\gamma(G) \leq \frac{5}{14}n$. In the light of these bounds on γ, in 2004 Seager considered bounds on d_g for r-regular graphs and showed that:

Theorem 1.1([10]) *For $r \geq 3$, $d_g \leq \frac{r^2+4r+1}{(2r+1)^2} \times n$.*

Theorem 1.2([3]) *For any graph of order n, $\left\lceil \frac{n}{1+\Delta G} \right\rceil \leq \gamma(G)$.*

The authors of [7] studied domination number in Hamiltonian cubic graphs, and stated in it the following problem.

Problem 1.3 *What are the domination numbers of the Hamiltonian 4-regular graphs?*

The aim of this article is to study the domination number $\gamma(G)$, independent domination number $i(G)$, connected domination number $\gamma_c(G)$, total domination number $\gamma_t(G)$ and forcing domination number $f(G,\gamma)$ for 4-regular graphs and give a sharp value for the domination numbers of these graphs.

§2. Domination Number

In this section we obtain a sharp value for the domination number of some 4-regular graph. In the following, we construct graphs G, G_1 and G_2 of which the graphs G and G_2 are 4-regular. The graph G_1 is not 4-regular but $deg_{G_1}(v_i) = 4$ where $2 \leq i \leq m-1$ and for the two remaining vertices, $deg_{G_1}(v_1) = deg_{G_1}(v_m) = 3$. Moreover, the graph G_2 will be obtained from the graphs G_1.

Remark 2.1 (i) Let G be a graph with $V(G) = \{v_1, v_2, ..., v_n\}$ and $E(G) = \{v_iv_j \mid |j-i| =$

$1 \ or \ t \ or \ t+1\} \cup \{v_1v_t, v_{t+1}v_{2t}\}$ where $n = 2t$, $t \geq 3$;

(ii) Let G_1 be a graph with $V(G_1) = \{v_1, v_2, ..., v_m\}$ and $E(G_1) = \{v_iv_j \mid |j-i| = 1 \ or \ s \ or \ s+1\}$ where $m = 2s+1$, $s \geq 2$;

(iii) Let $G_2 = \cup_{i=1}^{q} G_{m_i}$ where $G_{m_i} \cong G_1$, $|V(G_{m_i})| = m_i$ for all possible i and $|V(G_{m_1})| \leq |V(G_{m_2})| \leq ... \leq |V(G_{m_q})|$, such that $V(G_2) = \cup_{i=1}^{q} \cup_{j=1}^{m_i} \{v_iv_j\}$ and $E(G_2) = \cup_{i=1}^{q} E(G_{m_i}) \cup \{v_{im_i}v_{(i+1)1 \ (mod \ q)} | \ i = 1, 2, ..., q\}$.

By Theorem 1.1, we have $d_g \leq (33/81)n$ for r-regular graphs where $r = 4$. In the following Theorems, we obtain the exact number for constructed 4-regular graphs.

In all following theorems, let m, n be odd and even respectively and $n \equiv l_1 \pmod 5$, $m \equiv l_2 \pmod 5$ then $m = 5p + l_2$ and $n = 5k + l_1$ where $0 \leq l_1, l_2 \leq 4$ and p, k are integers.

By Theorem 1.2, we have the following observation.

Observation 2.2 $\gamma(G) \geq \lceil \frac{5k+l_1}{5} \rceil$ *and* $\gamma(G_1) \geq \lceil \frac{5p+l_2}{5} \rceil$.

Theorem 2.3 *Let G be a graph of order n, then* $\gamma(G) = \begin{cases} k & if \ n \equiv 0 \ (mod \ 5) \\ k+1 & otherwise \end{cases}$.

Proof We proceed by proving the series cases of following.

Case 1 If $n \equiv 0 \pmod 5$ then $n = 5k$. Let $S = \{v_3, v_8, v_{13}, \cdots, v_i, v_{i+5}, \cdots, v_{\frac{n}{2}-7}, v_{\frac{n}{2}-2}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+6}, \cdots, v_j, v_{j+5}, \cdots, v_{n-4}\}$. It is easy to verify that $|S| = (2 \times (\frac{n}{2} - 5)/5) + 2 = k$. Furthermore, every vertex in S dominates four vertices and itself and $N[x] \cap N[y] = \emptyset$ for any pair of vertices $x, y \in S$. It follows that S is a dominating set, so $\gamma(G) \leq k$. Using Observation 2.2 it is now straightforward to see that $\gamma(G) = k$.

Case 2 If $n \equiv 1 \pmod 5$ then $n = 5k+1$. Let $S = \{v_3, v_8, v_{13}, ..., v_i, v_{i+5}, ...v_{\frac{n}{2}-5}, v_{\frac{n}{2}-1}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+6}, \cdots, v_j, v_{j+5}, ..., v_{n-2}\}$ which implies $|S| = k+1$. Clearly, every vertex in $S - \{v_{\frac{n}{2}-1}\}$ dominates four vertices and itself. Then the non-dominated vertex $v_{\frac{n}{2}-1}$ is dominated by itself. Also, $N[x] \cap N[y] = \emptyset$ for every pair vertices $x, y \in S - \{v_{\frac{n}{2}-1}\}$. Thus S is a dominating set and $\gamma(G) \leq k+1$. Using Observation 2.2 it is now straightforward to see that $\gamma(G) = k+1$.

Case 3 If $n \equiv 2 \pmod 5$ so $n = 5k+2$. Assign $S = \{v_2, v_7, \cdots, v_i, v_{i+5}, \cdots, v_{\frac{n}{2}-9}, v_{\frac{n}{2}-4}, v_m, v_{\frac{n}{2}+5}, \cdots, v_j, v_{j+5}, ..., v_{n-1}\}$ and $m \in \{\frac{n}{2}, \frac{n}{2}+1\}$. One can see that any vertex in $S - \{v_m\}$ dominates four vertices and itself and the two non-dominated vertices $v_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$ are dominated by vertex v_m. Obviously, $|S| = k+1$. Moreover, for every pair of vertices x and y from $S - \{v_m\}$, we have $N[x] \cap N[y] = \emptyset$. Therefore S is a dominating set for G that implies $\gamma(G) \leq k+1$. Using Observation 2.2 it is now straightforward to see that $\gamma(G) = k+1$.

Case 4 If $n \equiv 3 \pmod 5$ so $n = 5k+3$. Let $S = \{v_2, v_7, v_{12}, \cdots, v_i, v_{i+5}, \cdots, v_{\frac{n}{2}-2}, v_m, v_{\frac{n}{2}+5}, \cdots, v_j, v_{j+5}, \cdots, v_{n-4}\}$, where $m \in \{\frac{n}{2}, \frac{n}{2}+1, n\}$. By simple verification one can see that every vertex in $S - \{v_m\}$ dominates four vertices and itself and the three vertices $v_{\frac{n}{2}}$, $v_{\frac{n}{2}+1}$ and v_n are dominated by vertex v_m. Clearly, $|S| = k+1$ and $N[x] \cap N[y] = \emptyset$ for all possible vertices $x, y \in S - \{v_m\}$. Therefore S is a dominating set for G that implies $\gamma(G) \leq k+1$. Using Observation 2.2 it is now straightforward to see that $\gamma(G) = k+1$.

Case 5 If $n \equiv 4 \pmod 5$, so $n = 5k+4$. Let $S = \{v_2, v_7, ..., v_i, v_{i+5}, ..., v_{\frac{n}{2}-5}, v_{\frac{n}{2}}, v_{\frac{n}{2}+5}, ..., v_j,$

$v_{j+5}, \cdots, v_{n-2}\}$. We see every vertex in $S - \{v_{\frac{n}{2}}\}$ dominated four vertices and itself and the vertex $v_{\frac{n}{2}}$ dominates three vertices $\{v_{\frac{n}{2}-1}, v_{\frac{n}{2}+1}, v_n\}$ and itself. Since $|S| = k+1$ and $N[x] \cap N[y] = \emptyset$ for all possible vertices $x, y \in S - \{v_{\frac{n}{2}}\}$. Then S is a dominating set for G that implies $\gamma(G) \leq k+1$. By Observation 2.2 it is straightforward to see that $\gamma(G) = k+1$. $\square$

Theorem 2.4 *Let G_1 be a graph of order $m = 5p + l_2$ where $l_2 \in \{0, 1, 2, 3, 4\}$ and p is an integer, then* $\gamma(G_1) = \begin{cases} p & if\ m \equiv 0\ (mod\ 5); \\ p+1 & otherwise. \end{cases}$

Proof We consider the following sets such that $m \equiv l_2$ (mod 5) for $0 \leq l_2 \leq 4$.

For $l_2 = 0$. We say $S = \{v_2, v_7, ..., v_i, v_{i+5}, ..., v_s, v_{s+5}, ..., v_j, v_{j+5}, ..., v_{m-3}\}$.
For $l_2 = 1$. We say $S = \{v_2, v_7, ..., v_i, v_{i+5}, ..., v_{s-3}, v_s, v_{s+5}, v_{s+10}, ..., v_j, v_{j+5}, ..., v_{m-1}\}$.
For $l_2 = 2$. We say $S = \{v_2, v_7, .., v_i, v_{i+5}, .., v_{s-1}, v_{s+1}, v_{s+5}, v_{s+10}, .., v_j, v_{j+5}, ..., v_{m-4}\}$.
For $l_2 = 3$. We say $S = \{v_2, v_7, ..., v_i, v_{i+5}, ..., v_{s-4}, v_s, v_{s+5}, v_{s+10}, ..., v_j, v_{j+5}, ..., v_{m-2}\}$.
For $l_2 = 4$. We say $S = \{v_2, v_7, ..., v_i, v_{i+5}, ..., v_{s-2}, v_s, v_{s+5}, v_{s+10}, ..., v_j, v_{j+5}, ..., v_{m-5}\}$.

A method similar to that described in proof of Theorem 2.3 can be applied for proof of this Theorem. From this, one can see that all of the considered sets are dominating sets. Using Observation 2.2 it is now straightforward to obtain the stated results in this Theorem. $\square$

Now we are ready to study domination number of more 4-regular graphs which are stated in Theorem 2.3.

Remark 2.5 We construct graph $G' = G_{n_1} \cup G_{n_2} \cup ... \cup G_{n_r}$ in which between every two 4-regular graphs we add an edge such that $d(v) = 5$ to each of first and end vertices of G_{n_i} for all possible i and $G_{n_i} \cong G$, $|V(G_{n_1})| \leq |V(G_{n_2})| \leq ... \leq |V(G_{n_r})|$.

Theorem 2.6 $\gamma(G') = \sum_{i=1}^{r} \gamma(G_{n_i})$ *such that there exists a G with $G_{n_i} \cong G$ for each i.*

Proof The result follows by Theorem 2.3. $\square$

Let $G_1^{'}$ and $G_1^{''}$ be the graphs in which these are two induced subgraphs of G_1 such that $V(G_1^{'}) = V(G_1) - \{v_1, v_m\}$ and $V(G_1^{''}) = V(G_1) - \{v_1\}$ (or $V(G_1^{''}) = V(G_1) - \{v_m\}$).

Proposition 2.7 *(i) $\gamma(G_1^{'}) = \gamma(G_1^{''}) = \gamma(G_1)$ where $V(G_1) \equiv l$ to modulo 5 and $l \in \{0, 2, 3, 4\}$;*

(ii)Let $V(G_1) \equiv 1$ to modulo 5. Then (a): $\gamma(G_1^{'}) = \gamma(G_1) - 1$ (b): $\gamma(G_1^{''}) = \gamma(G_1)$ where $V(G_1^{''}) = V(G_1) - \{v_1\}$ (c): $\gamma(G_1^{''}) = \gamma(G_1) - 1$ where $V(G_1^{''}) = V(G_1) - \{v_m\}$.

Proof (i) The result follows by Observation 2.2 and Theorem 2.4.

(ii) Let $V(G_1) \equiv 1$ (mod 5). We say $S = \{v_4, v_9, ..., v_{s-1}, v_{s+2}, v_{s+7}, ..., v_{m-4}\}$. Clearly S is a dominating set for $G_1^{'}$ and $G_1^{''}$ where $V(G_1^{''}) = V(G_1) - \{v_m\}$. Therefore $\gamma(G_1^{'}) = \gamma(G_1^{''}) = \gamma(G_1) - 1$ because $|S| = \frac{m-1}{5}$. Finally if $V(G_1^{''}) = V(G_1) - \{v_1\}$, one can check by simple verification that $\gamma(G_1^{''}) = \gamma(G_1)$. $\square$

Proposition 2.8 *Let G_2 be the graph with $V(G_{m_i}) \equiv 1$ (mod 5) for all i. Then* $\gamma(G_2) = \sum_{i=1}^{q} \gamma(G_{m_i}) - \lfloor \frac{q}{2} \rfloor$.

Proof The result follows by Proposition 2.7 (ii)(c). Moreover, It is sufficient to show the truth of the statement when $q = 2$ ($G_2 = G_{m_1} \cup G_{m_2}$).

Let $S = \{v_{14}, v_{19}, \cdots, v_{1i}, v_{1(i+5)}, \cdots, v_{1(s_{m_1}-1)}, v_{1(s_{m_1}+2)}, v_{1(s_{m_1}+7)}, \cdots, v_{1j}, v_{1(j+5)}, \cdots, v_{1(m_1-4)}, v_{21}, v_{26}, \cdots, v_{2i'}, v_{2(i'+5)}, \cdots, v_{2(s_{m_2}-4)}, v_{2(s_{m_2}+1)}, \cdots, v_{2j'}, v_{2(j'+5)}, \cdots, v_{2(m_2-2)}\}$.

Obviously, $\gamma(G_2) = \gamma(G_{m_1}) + \gamma(G_{m_2}) - 1$. It is now straightforward to prove the result for $q > 2$, by Proposition 2.7(ii) and a method similar to that described for $q = 2$. Thus $\gamma(G_2) = \sum_{i=1}^{q} \gamma(G_{m_i}) - \lfloor \frac{q}{2} \rfloor$ with $V(G_{m_i}) \equiv 1 \pmod 5$ for all i. □

Let l be the number of occurrences of consecutive G_1's with $V(G_1) \equiv 1 \pmod 5$. For $1 \leq i' \leq l$, let $H_{i'} = \{G_2 - e|\ G_2 = \cup_{j=1}^{r_{i'}} G_{m_j},\ r_{i'}$ is the number of consecutive $\mathrm{G_{m_j}}$s with $\mathrm{V(G_{m_j})} \equiv 1 \pmod 5$ for all $j,\ e (= v_{11}^{i'} v_{r_{i'} m_{r_{i'}}}^{i'}) \notin G_{m_j}\}$.

Theorem 2.9 *Let $G_3 = \cup_{i=1}^{q} G_{m_i}$ which contains the induced subgraph $H_{i'}$ for $1 \leq i' \leq l$ and $G_3 \cong G_2$. Then $\gamma(G_3) = \sum_{i=1}^{q} \gamma(G_{m_i}) - (\lfloor \frac{r_1}{2} \rfloor + \lfloor \frac{r_2}{2} \rfloor + ... + \lfloor \frac{r_l}{2} \rfloor)$.*

Proof The result follows by Theorem 2.4 and Propositions 2.7 and 2.8. □

§3. Independent Domination Number of Some Graphs

Theorem 3.1 *If $n \equiv l$ (mod 5) where $0 \leq l \leq 4$, then $i(G) = \gamma(G)$.*

Proof We Suppose that $n \equiv 0 \pmod 5$ and n is even. Since $i(G) \geq \gamma(G)$, Theorem 2.3 implies that $i(G) \geq k$. Let $S = S_1 \cup S_2 = \{v_3, ..., v_i, v_{i+5},\ ..., v_{\frac{n}{2}-7}, v_{\frac{n}{2}-2}\} \cup \{v_{\frac{n}{2}+1}, v_{\frac{n}{2}+6}, ..., v_j, v_{j+5}, ..., v_{n-4}\}$. It is sufficient to prove that there exists no pair of vertices (x, y) with $xy \in E(G)$ in S. Because, on the one hand $d_{P_n}(x, y) = 5$ (Let $P_n = v_1 v_2 ... v_n$) for any two consecutive vertices with $x, y \in S_1$ (or $x, y \in S_2$). On the other hand each $v_i \in S$ is adjacent to vertices v_{i-1}, v_{i+1}, $v_{i+\frac{n}{2}}$ and $v_{i+\frac{n}{2}+1}$. So by simple verification one can see that there exists no vertex in S from $\{v_{i-1}, v_{i+1}, v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1}\}$. Hence S is an independent set of G, then $i(G) = \gamma(G)$.

Similar argument settles proof of cases $n \equiv l$ where $1 \leq l \leq 4$. □

Theorem 3.2 *If $m \equiv l$ (mod 5) where $0 \leq l \leq 4$, then $i(G_1) = \gamma(G_1)$.*

Proof Similar to that of Theorem 3.1, we settle the proof of this Theorem. □

Theorem 3.3 *$i(G_x) = \gamma(G_x)$ where $x = 2, 3$.*

Proof The result follows by Theorems 2.4 and 2.9. □

§4. Connected Domination Number of Some Graphs

Let $N_p[v_i] = N[v_i] - (N(v_i) \cap S)$ where S is an arbitrary set.

Theorem 4.1 *If $n \equiv l$ (mod 5), where $0 \leq l \leq 4$, then $\gamma_c(G) = \frac{n}{2} - 1$.*

Proof Let $n \equiv 0 \pmod 5$. Since $\gamma_c(G) \geq \gamma(G)$, Theorem 2.3 implies $\gamma_c(G) \geq k$. We introduce $S_0 = \{v_2, v_3, ..., v_{\frac{n}{2}}\}$. Obviously, S_0 is a connected dominating set for G, then $\gamma_c(G) \leq \frac{n}{2} - 1$. Now we suppose that S is an arbitrary connected dominating set for G with $|S| = l \leq \frac{n}{2} - 2$. Clearly, $\langle S \rangle$ is containing a path of length $l \leq \frac{n}{2} - 2$, and $|N_p[x]|$, $|N_p[y]| \leq 4$ and $|N_p[z]| = 3$ where x, y are pendant vertices of path and $z \in S - \{x, y\}$. Furthermore $|N_p[u] \cap N_p[v]| = 1$ where u, v are two consecutive vertices from S. By the assumptions we have $|\cup_{x \in S} N_p[x]| \leq (2 \times 4) + (\frac{n}{2} - 4) \times 3 - (\frac{n}{2} - 3) = n - 1$. Then S cannot dominate all vertices of G. This implies that S_0 is minimum connected dominating set of G, hence $\gamma_c(G) = \frac{n}{2} - 1$.

Similar argument settles proof of cases $n \equiv l$ where $1 \leq l \leq 4$. □

Theorem 4.2 *If $m \equiv l$ (mod 5) where $0 \leq l \leq 4$ then $\gamma_c(G_1) = s - 1$.*

Proof In a manner similar to Theorem 4.1 we can prove the Theorem. □

Theorem 4.3 $\gamma_c(G_2) = \sum_{i=1}^{q}(s_{m_i} + 1) - 2$.

Proof Theorem 4.2 implies that $\gamma_c(G_2) \geq \sum_{i=1}^{q}(s_{m_i} - 1)$. Because if S_1 and S_2 are arbitrary γ_c-sets for G_{m_1} and G_{m_2} with $|S_1| = s_{m_1} - 1$, $|S_2| = s_{m_2} - 1$ then $\langle S_1 \cup S_2 \rangle$ is disconnected. Furthermore, any γ_c-set for G_{m_i} does not contain first or endvertex of G_{m_i}. Therefore, to obtain a γ_c-set for G_2, we must add all of the end and first vertices of the graph G_{m_i} except for two graphs. For the first graph, say (G_{m_1}), we can add the endvertex and the last graph, say (G_{m_q}), we may add its first vertex (note that we may choose in a similar manner for two other graphs). Then $\gamma_c(G_2) = \sum_{i=1}^{q}(s_{m_i} + 1) - 2$. □

§5. Total Domination Number of Some Graphs

Let S be a minimum total dominating set, then we have the following Observations.

Observation 5.1 For any vertex $x \in S$, there exists at least one vertex $y \in S$ such that $xy \in E(G)$.

Observation 5.2 Let G be a 4-regular graph then $|N[x] \cup N[y]| \leq 8$, where $x, y \in S$ and $xy \in E(G)$.

Immediately we have the following lemma.

Lemma 5.3 *Let G and G_1 be the graphs defined in Remark* 2.1. *For any $x, y \in S$ with $xy \in S$ then $|N[x] \cup N[y]| \leq 7$.*

Proof Let $x = v_i$ and $y = v_{i+1}$ (or $y = v_{i-1}$) then $|N[x] \cup N[y]| = 7$. Now suppose that $x = v_i$ and $y = v_{i+\frac{n}{2}}$ (or $y = v_{i+\frac{n}{2}+1}$) then $|N[x] \cup N[y]| = 6$. Hence $|N[x] \cup N[y]| \leq 7$. □

We consider the following Theorem.

Theorem 5.4 $\gamma_t(G) = \begin{cases} 2\lceil \frac{n}{7} \rceil & n \equiv l \ (mod\ 7) \ where\ l \in \{0, 3, 4, 5, 6\} \\ 2\lfloor \frac{n}{7} \rfloor + 1 & n \equiv 1 or\ 2\ (mod\ 7) \end{cases}$.

Proof The proof is divided into the following cases by considering $n \equiv$ (mod7).

Case 1 $n \equiv 0 \pmod 7$

Let $S = \{v_1, v_2, v_8, v_9, \cdots, v_i, v_{i+1}, \cdots, v_{\frac{n}{2}-6}, v_{\frac{n}{2}-5}, v_{\frac{n}{2}+5}, v_{\frac{n}{2}+6}, v_{\frac{n}{2}+12}, v_{\frac{n}{2}+13}, \cdots, v_j, v_{j+1}, \cdots, v_{n-2}, v_{n-1}\}$. It is easy to verify that S is a γ_t-set for G where $n \equiv 0 \pmod 7$. Moreover any two adjacent vertices from S have 7 vertices as neighbors, so by Lemma 5.3, S is minimum total dominating set for G and $\gamma_t(G) = |S| = 2\lceil \frac{n}{7} \rceil$ where $n \equiv 0 \pmod 7$.

Case 2 $n \equiv 1 \pmod 7$

Let $S_1 = \{v_1, v_2, v_8, v_9, \cdots, v_i, v_{i+1}, \cdots, v_{\frac{n}{2}-3}, v_{\frac{n}{2}-2}, v_{\frac{n}{2}+5}, v_{\frac{n}{2}+6}, v_{\frac{n}{2}+12}, v_{\frac{n}{2}+13}, \cdots, v_j, v_{j+1}, \cdots, v_{n-6}, v_{n-5}\}$. It is easy to verify that $(N[x] \cup N[y]) \cap (N[z] \cup N[t]) = \emptyset$ for each two pairs of vertices (x, y) and (z, t) and $xy, zt \in E(G)$ and $x, y, z, t \in S_1$. Also, $|N[r] \cup N[s]| = 7$ for all possible $r, s \in S_1$ and $rs \in E(G)$. Meanwhile, Lemma 5.3 implies that the set S_1 is a minimum γ_t-set for $G - M_1$ where $M_1 = \{v_n\}$. Now, we give $S_2 = \{v_{\frac{n}{2}-1}\}$. Clearly $S = S_1 \cup S_2$ is a γ_t-set of G where $n = 7k + 1$. Then $\gamma(G) = |S| = 2\lfloor \frac{n}{7} \rfloor + 1$ where $n \equiv 1 \pmod 7$.

Case 3 $n \equiv 2 \pmod 7$

Let $S_1 = \{v_1, v_2, v_8, v_9, \cdots, v_i, v_{i+1}, \cdots, v_{\frac{n}{2}-7}, v_{\frac{n}{2}-6}, v_{\frac{n}{2}+5}, v_{\frac{n}{2}+6}, v_{\frac{n}{2}+12}, v_{\frac{n}{2}+13}, \cdots, v_j, v_{j+1}, \cdots, v_{n-3}, v_{n-2}\}$. It is easy to verify that $(N[x] \cup N[y]) \cap (N[z] \cup N[t]) = \emptyset$ for each two pairs of vertices (x, y) and (z, t) with $xy, zt \in E(G)$ and $x, y, z, t \in S_1$. Also, $|N[r] \cup N[s]| = 7$ for all possible $r, s \in S_1$ and $rs \in E(G)$. Hence, Lemma 5.3 implies that the set S_1 is a minimum γ_t-set for $G - M_2$ where $M_2 = \{v_{\frac{n}{2}-1}, v_n\}$. Now, let $S_2 = \{v_{n-1}\}$. Clearly $S = S_1 \cup S_2$ is a γ_t-set of G where $n = 7k + 2$. Then $\gamma(G) = |S| = 2\lfloor \frac{n}{7} \rfloor + 1$ where $n \equiv 2 \pmod 7$.

Case 4 $n \equiv 3 \pmod 7$

Let $S_1 = \{v_1, v_2, v_8, v_9, ..., v_i, v_{i+1}, ..., v_{\frac{n}{2}-4}, v_{\frac{n}{2}-3}, v_{\frac{n}{2}+5}, v_{\frac{n}{2}+6}, v_{\frac{n}{2}+12}, v_{\frac{n}{2}+13}, \cdots, v_j, v_{j+1}, \cdots, v_{n-7}, v_{n-6}\}$. It is easy to verify that $(N[x] \cup N[y]) \cap (N[z] \cup N[t]) = \emptyset$ for each two pairs of vertices (x, y) and (z, t) with $xy, zt \in E(G)$ and $x, y, z, t \in S_1$. Furthermore, $|N[r] \cup N[s]| = 7$ for all possible $r, s \in S_1$ and $rs \in E(G)$. Clearly, Lemma 5.3 implies that the set S_1 is a minimum γ_t-set for $G - M_3$ where $M_3 = \{v_{\frac{n}{2}-1}, v_{n-1}, v_n\}$. Now, let S_2 be 2-subset from M_3 which are adjacent in G. Clearly $S = S_1 \cup S_2$ is a γ_t-set of G where $n = 7k + 3$. Then $\gamma(G) = |S| = 2\lceil \frac{n}{7} \rceil$ where $n \equiv 3 \pmod 7$.

Case 5 $n \equiv 4 \pmod 7$

We assign $S_1 = \{v_1, v_2, v_8, v_9, \cdots, v_i, v_{i+1}, \cdots, v_{\frac{n}{2}-8}, v_{\frac{n}{2}-7}, v_{\frac{n}{2}+5}, v_{\frac{n}{2}+6}, v_{\frac{n}{2}+12}, v_{\frac{n}{2}+13}, \cdots, v_j, v_{j+1}, ..., v_{n-4}, v_{n-3}\}$. It is easy to verify that $(N[x] \cup N[y]) \cap (N[z] \cup N[t]) = \emptyset$ for each two pairs of vertices (x, y) and (z, t) with $xy, zt \in E(G)$. Also, $|N[r] \cup N[s]| = 7$ for all possible $r, s \in S_1$ and $rs \in E(G)$. Hence, Lemma 5.3 implies that the set S_1 is a minimum γ_t-set for $G - M_4$, where $M_4 = \{v_{\frac{n}{2}-2}, v_{\frac{n}{2}-1}, v_{n-1}, v_n\}$. Now, let S_2 be a 2-subset from M_4 which are adjacent in G. Clearly $S = S_1 \cup S_2$ is a γ_t-set of G where $n = 7k + 4$. Then $\gamma(G) = |S| = 2\lceil \frac{n}{7} \rceil$ where $n \equiv 4 \pmod 7$.

Case 6 $n \equiv 5 \pmod 7$

Say $S_1 = \{v_1, v_2, v_8, v_9, \cdots, v_i, v_{i+1}, \cdots, v_{\frac{n}{2}-5}, v_{\frac{n}{2}-4}, v_{\frac{n}{2}+5}, v_{\frac{n}{2}+6}, v_{\frac{n}{2}+12}, v_{\frac{n}{2}+13}, \cdots, v_j, v_{j+1},$

$\cdots, v_{n-8}, v_{n-7}\}$. It is easy to verify that $(N[x] \cup N[y]) \cap (N[z] \cup N[t]) = \emptyset$ for each two pairs of vertices (x, y) and (z, t) with $xy, zt \in E(G)$ and $x, y, z, t \in S_1$. Also, $|N[r] \cup N[s]| = 7$ for all possible $r, s \in S_1$ and $rs \in E(G)$. Hence, Lemma 5.3 implies that the set S_1 is a minimum γ_t-set for $G - M_5$, where $M_5 = \{v_{\frac{n}{2}-2}, v_{\frac{n}{2}-1}, v_{n-2}, v_{n-1}, v_n\}$. Now, let $S_2 = \{v_{\frac{n}{2}-2}, v_{\frac{n}{2}-1}\}$. Clearly $S = S_1 \cup S_2$ is a γ_t-set of G where $n = 7k + 5$. Then $\gamma(G) = |S| = 2\lceil \frac{n}{7} \rceil$ where $n \equiv 5 \pmod 7$.

Case 7 $n \equiv 6 \pmod 7$

Let $S_1 = \{v_1, v_2, v_8, v_9, ..., v_i, v_{i+1}, \cdots, v_{\frac{n}{2}-9}, v_{\frac{n}{2}-8}, v_{\frac{n}{2}+5}, v_{\frac{n}{2}+6}, v_{\frac{n}{2}+12}, v_{\frac{n}{2}+13}, \cdots, v_j, v_{j+1}, \cdots, v_{n-5}, v_{n-4}\}$. It is easy to verify that $(N[x] \cup N[y]) \cap (N[z] \cup N[t]) = \emptyset$ for each two pairs of vertices (x, y) and (z, t) with $xy, zt \in E(G)$ and $x, y, z, t \in S_1$. Also, $|N[r] \cup N[s]| = 7$ for all possible $r, s \in S_1$ and $rs \in E(G)$. Hence, Lemma 5.3 implies that the set S_1 is a minimum γ_t-set for $G - M_6$, where $M_6 = \{v_{\frac{n}{2}-3}, v_{\frac{n}{2}-2}, v_{\frac{n}{2}-1}, v_{n-2}, v_{n-1}, v_n\}$. Now, let $S_2 = \{v_{\frac{n}{2}-2}, v_{\frac{n}{2}-1}\}$. Clearly $S = S_1 \cup S_2$ is a γ_t-set of G where $n = 7k + 6$. Then $\gamma(G) = |S| = 2\lceil \frac{n}{7} \rceil$ where $n \equiv 6 \pmod 7$. □

Theorem 5.5 $\gamma_t(G_1) = \begin{cases} 2\lceil \frac{m}{7} \rceil & if\ m \equiv l\ (mod\ 7)\ where\ l \in \{0, 3, 4, 5, 6\} \\ 2\lfloor \frac{m}{7} \rfloor + 1 & if\ m \equiv 1\ or\ 2\ (mod\ 7) \end{cases}$.

Proof Lemma 5.3 implies that $\gamma_t(G_1) \geq 2\lceil \frac{m}{7} \rceil$. Now we consider the following cases.

Case 1 $m \equiv 0 \pmod 7$

We assign $S_{t_0} = \{v_5, v_6, v_{12}, v_{13}, \cdots, v_i, v_{i+1}, \cdots, v_{s-5}, v_{s-4}, v_{s+2}, v_{s+3}, \cdots, v_j, v_{j+1}, \cdots, v_{m-2}, v_{m-1}\}$. It is easy to see that S_{t_0} is a γ_t-set for G_1. Hence $\gamma_t(G_1) \leq 2\lceil \frac{m}{7} \rceil$. Moreover Lemma 5.3 implies $\gamma_t(G_1) \geq 2\lceil \frac{m}{7} \rceil$. It follows that $\gamma_t(G_1) = 2\lceil \frac{m}{7} \rceil$ with $m \equiv 0 \pmod 7$.

Case 2 $m \equiv l \pmod 7$ where $l \in \{1, 2, 3, 4, 5, 6\}$.

We assign S_{t_l} to each l as follows:

$S_{t_1} = \{v_1, v_2, v_3, v_9, v_{10}, ..., v_i, v_{i+1}, v_{i+7}, ..., v_{s-5}, v_{s-4}, v_{s+6}, v_{s+7}, ..., v_j, v_{j+1}, ..., v_{m-2}, v_{m-1}\}$.

$S_{t_2} = \{v_2, v_3, v_9, v_{10}, ..., v_i, v_{i+1}, v_{i+7}, ..., v_{s-2}, v_{s-1}, v_s, v_{s+6}, v_{s+7}, ..., v_j, v_{j+1}, ..., v_{m-6}, v_{m-5}\}$.

$S_{t_3} = \{v_3, v_4, v_{10}, v_{11}, ..., v_i, v_{i+1}, v_{i+7}, ..., v_{s-5}, v_{s-4}, v_s, v_{s+1}, v_{s+7}, v_{s+8}, ..., v_j, v_{j+1}, ..., v_{m-2}, v_{m-1}\}$.

$S_{t_4} = \{v_1, v_2, v_7, v_8, ..., v_i, v_{i+1}, v_{i+7}, .., v_{s-5}, v_{s-4}, v_{s+4}, v_{s+5}, .., v_j, v_{j+1}, ..., v_{m-2}, v_{m-1}\}$.

$S_{t_5} = \{v_4, v_5, v_{11}, v_{12}, v_{18}, .., v_i, v_{i+1}, v_{i+7}, .., v_{s-5}, v_{s-4}, v_{s+1}, v_{s+2}, v_{s+8}, v_{s+9}, ..., v_j, v_{j+1}, ..., v_{m-2}, v_{m-1}\}$.

$S_{t_6} = \{v_1, v_2, v_8, v_9, .., v_i, v_{i+1}, v_{i+7}, .., v_{s-5}, v_{s-4}, v_{s+5}, v_{s+6}, .., v_j, v_{j+1} ..., v_{m-2}, v_{m-1}\}$.

In the same manner as in Case 1 we settle this Case. Hence $\gamma_t(G_1) = 2\lceil \frac{m}{7} \rceil$ where $m \equiv 3$ or 4 or 5 or 6 (mod 7) and $\gamma_t(G_1) = 2\lfloor \frac{m}{7} \rfloor + 1$ where $m \equiv 1$ or 2 (mod 7). □

Motivated by Theorem 5.5, we are now really ready to state of following Theorem.

Theorem 5.6 $\gamma_t(G_2) = \sum_{i=1}^{q} \gamma_t(G_{m_i})$.

§6. Forcing Domination Number of Some Graphs

Observation 6.1 $f(H,\gamma) \geq 1$ where $H \in \{G, G_1, G_2\}$.

Proof It is easy to see that the graphs G, G_1 and G_2 have at least two γ-sets. Then it immediately implies that $f(H,\gamma) \geq 1$ where $H \in \{G, G_1, G_2\}$. □

Observation 6.2 *$f(G,\gamma), f(G_1,\gamma) \geq 2$ where $|V(G)|, |V(G_1)| \equiv l$ with $l \in \{1,2,3,4\}$ (mod 5).*

Proof It is straightforward to see that with any 1-subset, say T from any arbitrary dominating set, we can obtain at least two different γ-sets for G containing T. Then $f(G,\gamma) \geq 2$.

Similar argument settles that $f(G_1,\gamma) \geq 2$ too. □

Theorem 6.3 *(i) If $n \equiv 0$ (mod 5) then $f(G,\gamma) = 1$;*

(ii)If $m \equiv 0$ (mod 5) then $f(G_1,\gamma) = 1$;

(iii) $f(G_2,\gamma) = q$ where $V(G_{m_i}) \equiv 0$ (mod 5) for all i.

Proof (i) We apply Observation 6.1 with $H = G$, so $f(G,\gamma) \geq 1$. Now let $S = \{v_3, v_8, ..., v_i, v_{i+5}, \cdots, v_{\frac{n}{2}-2}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+6}, \cdots, v_j, v_{j+5}, \cdots, v_{n-4}\}$. It is easy to see that $F = \{v_3\} \subset S$ is a forcing subset for G which implies $f(G,\gamma) \leq 1$. It is now straightforward to give $f(G,\gamma) = 1$.

(ii) By Observation 6.1 with $H = G_1$, it implies that $f(G_1,\gamma) \geq 1$. Let $F = \{v_2\}$. Obviously, F is a forcing subset for G_1. From this and by Theorem 2.4, it follows that $f(G_1,\gamma) = 1$.

(iii) The Case(ii) settles this case. Moreover, let $F = \{v_{12}, v_{22}, v_{32}, ..., v_{i2}, ..., v_{q2}\}$ then it implies that $f(G_2,\gamma) = q$. □

Theorem 6.4 *(i) If $n \equiv 1$ (mod 5), then $f(G,\gamma) = 2$;*

(ii)If $m \equiv 1$ (mod 5), then $f(G_1,\gamma) = 2$;

(iii) $f(G_2,\gamma) = 2\lceil \frac{q}{2} \rceil$ where $V(G_{m_i}) \equiv 1$ (mod 5) for all i.

Proof (i) Observation 6.2 implies that $f(G,\gamma) \geq 2$. Say $S = \{v_1, v_6, ..., v_i, v_{i+5}, \cdots, v_{\frac{n}{2}-2}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+6}, \cdots, v_j, v_{j+5}, \cdots, v_{n-4}\}$. Suppose that $F = \{v_1, v_{\frac{n}{2}+1}\} \subset S$, clearly F is a forcing subset for G and it follows that $f(G,\gamma) \leq 2$. This implies that $f(G,\gamma) = 2$.

(ii)Using Observation 6.2 we have $f(G_1,\gamma) \geq 2$. Now we define $F = \{v_s, v_{m-1}\}$. Clearly, $|N[v_s] \cup N[v_{m-1}]| = 6$. On the other hand, since $m \equiv 1$ (mod 5) then cardinality of the set of remaining vertices is multiple of 5. It immediately follows that the set $\{v_2, v_7, ..., v_i, v_{i+5}, ..., v_{s-3}, v_{s+5}, v_{s+10}, ..., v_j, v_{j+5}, ..., v_{m-6}\} \cup F$ is the unique γ-set containing F. Thus $f(G_1,\gamma) = 2$.

(iii) We consider the following cases. (a): If q is even, let $F_1 = \cup_{i=2}^{q}\{v_{i1}, v_{i(s_i+1)}\}$ where i is even. (b): If q is odd let $F_2 = \cup_{i=2}^{q-1}\{v_{i1}, v_{i(s_i+1)}\} \cup \{v_{(q)1}, v_{q(s_q+1)}\}$ where i is even. By simple verification one can check that F_1 and F_2 are forcing subsets for G_2 in two stated cases. Hence, it follows that $f(G_2,\gamma) = 2\lceil \frac{q}{2} \rceil$. □

Theorem 6.5 *(i) If $n \equiv 2$ (mod 5), then $f(G,\gamma) = 2$;*

(ii) If $m \equiv 2$ (mod 5), then $f(G_1,\gamma) = 2$;

(iii) $f(G_2,\gamma) = 2q$ where $V(G_{m_i}) \equiv 2$ (mod 5) for all i.

Proof (i)Using Observation 6.2 we have $f(G,\gamma) \geq 2$. Now we define $F = \{v_{\frac{n}{2}-1}, v_{\frac{n}{2}}\} \subset S$. Clearly, $|N[v_{\frac{n}{2}-1}] \cup N[v_{\frac{n}{2}}]| = 7$. Moreover, since $m \equiv 2 \pmod 5$ then cardinality of the set of remaining vertices is a multiple of 5. It immediately follows that the set $\{v_5, v_{10}, ..., v_i, v_{i+5}, ..., v_{\frac{n}{2}-6}, v_{\frac{n}{2}+3}, v_{\frac{n}{2}+8}, ..., v_j, v_{j+5}, ..., v_{n-3}\} \cup F$ is the unique γ-set containing F. Thus $f(G,\gamma) = 2$.

(ii) Using Observation 6.2 we have $f(G_1,\gamma) \geq 2$. Now we define $F = \{v_{s+1}, v_{s+2}\}$. It immediately follows that the set $\{v_4, v_9, ..., v_i, v_{i+5}, ..., v_{s-4}, v_{s+7}, v_{s+12}, ..., v_j, v_{j+5}, ..., v_{m-2}\} \cup F$ is the unique γ-set containing F. Thus $f(G_1,\gamma) = 2$.

(iii): Clearly, the obtained forcing subset in the case (ii) is extendible to G_2. Therefore, we can assert that $f(G_2,\gamma) = 2q$. □

Theorem 6.6 *(i) If $n \equiv 3$ (mod 5), then $f(G,\gamma) = 2$;*

(ii) If $m \equiv 3$ (mod 5), then $f(G_1,\gamma) = 2$;

(iii) $f(G_2,\gamma) = 2q$ where $V(G_{m_i}) \equiv 3$ (mod 5) for all i.

Proof (i) Using Observation 6.2 we have $f(G,\gamma) \geq 2$. Now we define $F = \{v_1, v_{\frac{n}{2}+3}\} \subset S$. Clearly, $|N[v_1] \cup N[v_{\frac{n}{2}+3}]| = 8$. On the other hand, since $m \equiv 2 \pmod 5$ then cardinality of the set of remaining vertices is a multiple of 5. It immediately follows that the set $\{v_5, v_{10}, ..., v_i, v_{i+5}, ..., v_{\frac{n}{2}-4}, v_{\frac{n}{2}+8}, v_{\frac{n}{2}+13}, ..., v_j, v_{j+5}, ..., v_{n-1}\} \cup F$ is the unique γ-set containing F. Thus $f(G,\gamma) = 2$.

(ii) Using Observation 6.2 we have $f(G_1,\gamma) \geq 2$. Let $F = \{v_1, v_3\}$. Since $|N[v_1] \cup V[v_3]| = 8$, cardinality of the set of non-dominated vertices is a multiple of 5. From this it immediately follows that S consists of v_{s+6}, v_8, v_{s+11}, v_{13},...,v_{m-1}, v_{s-3}. Thus $f(G_1,\gamma) = 2$.

(iii) Clearly, the obtained forcing subset in Case (ii) is extendible to G_2. Therefore, it implies that $f(G_2,\gamma) = 2q$. □

Theorem 6.7 *(i) If $n \equiv 4$ (mod 5), then $f(G,\gamma) = 2$;*

(ii) If $m \equiv 4$ (mod 5), then $f(G_1,\gamma) = 2$;

(iii) $f(G_2,\gamma) = 2q$ where $V(G_{m_i}) \equiv 4$ (mod 5) for all i.

Proof (i) Using Observation 6.2 we have $f(G,\gamma) \geq 2$. Now we define $F = \{v_{\frac{n}{2}-2}, v_{\frac{n}{2}}\} \subset S$. Clearly, $|N[v_{\frac{n}{2}-2}] \cup N[v_{\frac{n}{2}}]| = 9$. Furthermore, since $m \equiv 2 \pmod 5$ then cardinality of the set of remaining vertices is a multiple of 5. It immediately follows that the set $\{v_5, v_{10}, \cdots, v_i, v_{i+5}, \cdots, v_{\frac{n}{2}-7},$
$v_{\frac{n}{2}+3}, v_{\frac{n}{2}+8}, \cdots, v_j, v_{j+5}, \cdots, v_{n-4}\} \cup F$ is the unique γ-set containing F. Thus $f(G,\gamma) = 2$.

(ii) By Observation 6.2 we have $f(G_1,\gamma) \geq 2$. Let $F = \{v_s, v_{s+2}\}$. It immediately follows that the set $\{v_4, v_9, \cdots, v_i, v_{i+5}, \cdots, v_{s-5}, v_{s+7}, v_{s+12}, \cdots, v_j, v_{j+5}, \cdots, v_{m-3}\} \cup F$ is the unique γ-set containing F. Thus $f(G_1,\gamma) = 2$.

(iii) Clearly, the obtained forcing subset in Case (ii) is extendible to G_2. Therefore, it implies that $f(G_2,\gamma) = 2q$. □

We close this section by the following Theorem for which we are motivated by the results of this section.

Theorem 6.8 *Let G_3 be the graph defined in Section 2. Then* $f(G_3,\gamma)=\sum_{i=1}^{q} f(G_{m_i},\gamma)-(\lfloor\frac{r_1}{2}\rfloor+\lfloor\frac{r_2}{2}\rfloor+...+\lfloor\frac{r_l}{2}\rfloor)$.

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Math.Combin.Book.Ser. Vol.4 (2009), 31-40

Computing Smarandachely Scattering Number of Total Graphs

Aysun AYTAC and Elgin KILIC

(Department of Mathematics of Ege University Faculty of Science, Bornova-IZMIR 35100, TURKEY)

E-mail: aysun.aytac@ege.edu.tr, elgin.kilic@ege.edu.tr

Abstract: Let S be a set consist of chosen components in $G-X$. The *Smarandachely scattering number* of a graph G is defined by

$$\gamma_S(G) = \max\{w(G-X) - |X| - \sum_{H\in S}|H| : X \subset V(G), w(G-X) > 1\}.$$

Particularly, if $S=\emptyset$ or $S=\{the\ \ largest\ \ component\ \ in\ \ G-X\}$, then $\gamma_S(G)$ is the scattering number or rupture degree of a graph G. In this paper, general results on the Smarandachely scattering number of a graph are considered. Firstly the relationships between the Smarandachely scattering number and some vulnerability parameters, namely scattering, integrity and toughness are given. Further, we calculate the Smarandachely scattering number of total graphs. Also several results are given about total graphs and graph operations.

Key Words: Smarandachely scattering number, connectivity, network design and communication, graph operations, rupture degree.

AMS(2000): 05C40, 05C76, 68M10, 68R10.

§1. Introduction

In a communication network, the vulnerability measures the resistance of network to disruption of operation after the failure of certain stations or communication links. The stability of communication networks is of prime importance to network designers.In analysis of vulnerability of a communication network to disruption, two quantities that come to mind are:

(1)the size of the largest remaining group within which mutual communication can still occur,

(2)the number of elements that are not functioning.

If we think of the graph as a model of a communication network, many graph theoreti-

[1]Received Sep.28, 2009. Accepted Oct. 18, 2009.

cal parameters have been used to describe the stability of communication networks including connectivity, integrity, toughness, tenacity, binding number and scattering number (see [2]-[3], [7]-[9] and [13]).

A graph G is denoted by $G = (V(G), E(G))$, where $V(G)$ is the vertex set of G and $E(G)$ is the edges set of G. The number of vertices and the number of edges of the graph G are denoted by $|V| = n$, $|E| = q$ respectively.

In this paper we will deal with the Smarandachely scattering number. But first we will give some basic definitions and notation. After that we give the the Smarandachely scattering number of total graph of specific families of graphs (see [4],[6],[10] and [12]).

•$t(G)$: The toughness of a graph G is defined by

$$t(G) = \min_{X \subseteq V(G)} \frac{|X|}{w\left(G - X\right)},$$

where X is a vertex cut of G and $w(G - X)$ is the number of the components of $G - X$.

•$I(G)$: The integrity of a graph is given by

$$I(G) = \min_{X \subseteq V(G)} \left\{|X| + m\left(G - X\right)\right\},$$

where $m(G - X)$ is the maximum number of vertices in a component of $G - X$.

•$s(G)$: The scattering number of a graph is defined by

$$s(G) = \max\left\{w(G - X) - |X| : X \subseteq V(G),\, w(G - X) \geqslant 2\right\},$$

where $w(G - X)$ denotes the number of components of the graph $G - X$.

•$\gamma_S(G)$: The *Smarandachely scattering number* of a graph G is defined by

$$\gamma_S(G) = \max\{w(G - X) - |X| - \sum_{H \in S} |H| : X \subset V(G), w(G - X) > 1\}.$$

Particularly, if $S = \emptyset$ or $S = \{\text{the largest component in } G - X\}$, then $\gamma_S(G)$ is the scattering number or rupture degree of a graph G (see [11]).

Definition 1.1 *Two vertices are said to cover each other in a graph G if they are incident in G. A vertex cover in G is a set of vertices that covers all edges of G. The minimum cardinality of a vertex cover in a graph G is called the vertex covering number of G and is denoted by $\alpha(G)$(see [4],[6],[10] and [12]).*

Definition 1.2([4],[6],[10] and [12]) *An independent set of vertices of a graph G is a set of vertices of G whose elements are pairwise nonadjacent. The independence number $\beta(G)$ of G is the maximum cardinality among all independent sets of vertices of G.*

Theorem 1.1([10],[12]) *For any graph G of order n,*

$$\alpha(G) + \beta(G) = n.$$

Definition 1.3 *The vertex-connectivity or simply connectivity $k(G)$ of a graph G is the minimum number of vertices whose removal from G result in a disconnected or trivial graph. The complete graph K_n cannot be disconnected by the removal of vertices, but the deletion of any n−1 vertices result in K_n; thus $k(K_n) = n-1$.*

$$k(G) = min\{|X| : X \subset V(G), \omega(G-X) > 1\},$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x. $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

§2. Some Results

We use P_n and C_n to denote the path and cycle with n vertices, respectively. A comet $C_{t,r}$ is defined as the graph obtained by identifying one end of the path $P_t, (t \geqslant 2)$ with the center of the star $K_{1,r}$. In this section we review the Smarandachely scattering number of P_n, C_n, $C_{t,r}$ and the k-complete partite graph $K_{n_1,n_2,\ldots,n_k}$.

Theorem 2.1([11]) *The Smarandachely scattering number of the comet $C_{t,r}$ the path $P_n, (n \geqslant 3)$, the star $K_{1,n-1}, (n \geqslant 3)$ and the cycle C_n are given in the following.*

a) *The Smarandachely scattering number of the comet $C_{t,r}$ is*

$$\gamma_S\left(C_{t,r}\right) = \begin{cases} r-1, & \text{if t is even} \\ r-2, & \text{if t is odd} \end{cases}$$

b) *The Smarandachely scattering number of the path P_n $(n \geqslant 3)$ is*

$$\gamma_S\left(P_n\right) = \begin{cases} -1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

c) *The Smarandachely scattering number of the star $K_{1,n-1}$ $(n \geqslant 3)$ is n-3.*

d) *The Smarandachely scattering number of the cycle C_n is*

$$\gamma_S\left(C_n\right) = \begin{cases} -1, & \text{if } n \text{ is even} \\ -2, & \text{if } n \text{ is odd} \end{cases}$$

Theorem 2.2([11]) *The Smarandachely scattering number of the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is $2\max\{n_1, n_2, \ldots, n_k\} - \sum\limits_{i=1}^{k} n_i - 1$.*

Theorem 2.3([11]) *Let G_1 and G_2 be two connected graphs of order n_1 and n_2, respectively. Then $\gamma_S(G_1 + G_2) = max\{\gamma_S(G_1) - n_2, \gamma_S(G_2) - n_1\}$.*

Theorem 2.4([11]) *Let G be an incomplete connected graph of order n. Then*

a) $2\alpha(G) - n - 1 \leqslant \gamma_S(G) \leqslant \frac{[\alpha(G)]^2 - \kappa(G)[\alpha(G)-1]-n}{\alpha(G)}$.

b) $3 - n \leqslant \gamma_S(G) \leqslant n - 3$.

c) $\gamma_S(G) \leqslant 2\delta(G) - 1$.

§3. Bounds for Smarandachely Scattering Number

In this section, we consider the relations between the Smarandachely scattering number and toughness, integrity and scattering number. Parameters that will be used in this paper are as the following:

$\alpha(G)$, the covering number;

$\beta(G)$, the independence number;

$k(G)$, the connectivity number;

$\delta(G)$, the minimum vertex degree and

$\Delta(G)$, the maximum vertex degree.

Theorem 3.1 *Let G be a connected graph of order n such that $t(G) = t$, $\gamma_S(G) = \gamma_S$ and $\delta(G) = \delta$. Then $\gamma_S \leqslant \frac{n}{t+1} - (\delta + 1)$.*

Proof Let X be cut set of vertices of G. From the definition of $t(G)$, we know that $t \leqslant \frac{|X|}{w(G-X)}$. Therefore,

$$w(G - X) \leqslant \frac{|X|}{t}.$$

We also have $w(G - X) + |X| \leq n$. In this inequality, $|X| \leq n - w(G - X)$and get $w(G - X) + |X| \leqslant n$. In this inequality, $|X| \leqslant n - w(G - X)$. Therefore,

$w(G - X)$	$\leqslant \frac{\lvert X\rvert}{t}$
$w(G - X)$	$\leqslant \frac{n-w(G-X)}{t}$
$w(G - X)$	$\leqslant \frac{n}{t+1}$.

On the other hand, for every graph G, it's known that

$$\delta(G) + 1 \leqslant I(G) \leqslant \alpha(G) + 1$$

and

$$I(G) = |X| + m(G - X) \geqslant \delta(G) + 1.$$

Then, we have $m(G - X) \geqslant \delta(G - X) + 1 \geqslant \delta(G) - |X| + 1$.
Therefore, we have $m(G - X) \geqslant \delta(G) + 1 - |X|$.

Let's construct the definition of the Smarandachely scattering number.

$w(G-X)-\|X\|-m(G-X)$	$\leqslant w(G-X)-\|X\|-\delta(G)-1+\|X\|$
$\gamma_S(G)$	$\leqslant w(G-X)-\delta(G)-1$
$\gamma_S(G)$	$\leqslant \frac{n}{t+1}-\delta(G)-1$

The proof is completed. $\square$

Theorem 3.2 *Let G be a connected graph of order n such that $t(G)=t$, $\gamma_S(G)=\gamma_S$, $\alpha(G)=\alpha$ and $k(G)=k$. Then $\gamma_S(G)\geqslant \frac{k}{t}-(\alpha+1)$.*

Proof Let X be a cut set of vertices of G. From the definition of $\gamma_S(G)$, we know that $w(G-X)-|X|-m(G-X)\leqslant \gamma_S$. Moreover, for every graph G, it is known that $I(G)\leqslant \alpha(G)+1$. So, we have $I(G)=|X|+m(G-X)\leqslant \alpha(G)+1$. We have the following inequality:

$$w(G-X)\leqslant \gamma_S(G)+\alpha(G)+1.$$

$\frac{1}{w(G-X)}$	$\geqslant \frac{1}{\gamma_S+\alpha+1}$
$\frac{\|X\|}{w(G-X)}$	$\geqslant \frac{\|X\|}{\gamma_S+\alpha+1},\quad \|X\|\geqslant k(G)$
$\frac{\|X\|}{w(G-X)}$	$\geqslant \frac{k}{\gamma_S+\alpha+1}$
$min\left\{\frac{\|X\|}{w(G-X)}\right\}$	$\geqslant min\left\{\frac{k}{\gamma_S+\alpha+1}\right\}$
t	$\geqslant \frac{k}{\gamma_S+\alpha+1}$
γ_S	$\geqslant \frac{k}{t}-(\alpha+1)$

The proof is completed. $\square$

Theorem 3.3 *Let$G$ be a non-complete connected graph such that $s(G)=s$, $\gamma_S(G)=\gamma_S$, $I(G)=I$ and $\alpha(G)=\alpha$ is the covering number of graph G. Then $\gamma_S \leqslant s$ - $I+\alpha$.*

Proof Let X be a vertex cut of G, then from the definition of $s(G)$ we know that $w(G-X)-|X| \leqslant s$.

When we subtract $m(G-X)$ from both sides of this inequality, we have the following.

$$w(G-X)-|X|-m(G-X)\leqslant s-m(G-X).$$

From the definition of $I(G)$ we know that $I(G)\leqslant |X|+m(G-X)$.

$$I(G)\leqslant |X|+m(G-X)\Rightarrow \quad m(G-X)\geqslant I-|X|$$
$$-m(G-X)\leqslant -I+|X|.$$

Then we have,

$$w(G-X)-|X|-m(G-X)\leqslant s-I+|X|$$

since X is a cut set of vertices, $|X|\leqslant \alpha$ is always satisfied,

$$w(G-X)-|X|-m(G-X)\leqslant s-I+\alpha$$

$$\max\{w(G-X)-|X|-m(G-X)\}\leqslant\max\{s-I+\alpha\}$$
$$r\leqslant s-I+\alpha$$

The proof is completed. □

§4. The Smarandachely Scattering Number of of Total Graphs Some Graph Types and Cartesian Product of Graphs

In this section, firstly, we will give definition of total graph of a graph and Cartesian product operation on graphs. After that we will give some results about the The Smarandachely scattering number of $T(P_n)$, $T(C_n), T(S_{1,n}), T(K_2xP_n)$ and $T(K_2xC_n)$.

Definition 4.1 *The vertices and edges of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. The total graph $T(G)$ of the graph $G=(V(G),E(G))$, has vertex set $V(G)\cup E(G)$, and two vertices of $T(G)$ are adjacent whenever they are neighbors in G. It is easy to see that $T(G)$always contains both G and Line graph $L(G)$ as a induced subgraphs. The total graph is the largest graph that is formed by the adjacent relations of elements of a graph.*

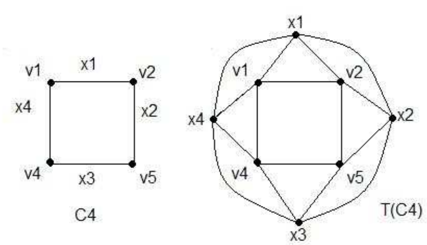

Fig.1

Definition 4.2 *The Cartesian product of two graphs G_1 and G_2, denoted by G_1xG_2, is defined as follows:*

$V(G_1xG_2)=V(G_1)xV(G_2)$, two vertices (u_1,u_2) and (v_1,v_2) are adjacent if and only if $u_1=v_1$ and $u_2v_2\in E(G_2)$ or $u_1v_1\in E(G_1)$ and $u_2=v_2$. The Cartesian product of n graphs $G_1,G_2,\cdots,G_n$ denoted by $G_1xG_2x\cdots xG_n$ is defined inductively as the Cartesian product $G_1xG_2x\cdots xG_{n-1}$ and G_n.

Theorem 4.1 *The Smarandachely scattering number of $T(P_n)$ order of 2n-1 is*

$$\gamma_S(T(P_n))=2-\left\lfloor\sqrt{1+2n}\right\rfloor-\left\lceil\frac{1+2n-2\left\lfloor\sqrt{1+2n}\right\rfloor}{\left\lfloor\sqrt{1+2n}\right\rfloor}\right\rceil.$$

Proof If we remove p vertices from graph $T(P_n)$, then the number of the remaining connected components is at most $\left\lfloor \frac{p}{2} \right\rfloor + 1$. In this case the order of the largest remaining component is $m(G-X) \geqslant \frac{2n-1-p}{\left\lfloor \frac{p}{2} \right\rfloor + 1}$. So,

$$\gamma_S(T(P_n)) \quad \geqslant max_p \left\{ \frac{p}{2} + 1 - p - \frac{2n-1-p}{\frac{p}{2}+1} \right\}.$$

The function $\frac{p}{2} + 1 - p - \frac{2n-1-p}{\frac{p}{2}+1}$ takes its maximum value at $p = -2 + 2\sqrt{1+2n}$. Then we write this p value in the definitions of w and m to calculate the Smarandachely scattering number as follows:

$$w = \left\lfloor \sqrt{1+2n} \right\rfloor, \quad m = \left\lceil \frac{1+2n-2\left\lfloor \sqrt{1+2n} \right\rfloor}{\left\lfloor \sqrt{1+2n} \right\rfloor} \right\rceil,$$

$$\gamma_S = \left\lfloor \sqrt{1+2n} \right\rfloor \; - \left(-2 + 2\left\lfloor \sqrt{1+2n} \right\rfloor \right) - \left\lceil \frac{1+2n-2\left\lfloor \sqrt{1+2n} \right\rfloor}{\left\lfloor \sqrt{1+2n} \right\rfloor} \right\rceil$$

and

$$\gamma_S(T(P_n)) = 2 - \left\lfloor \sqrt{1+2n} \right\rfloor \; - \left\lceil \frac{1+2n-2\left\lfloor \sqrt{1+2n} \right\rfloor}{\left\lfloor \sqrt{1+2n} \right\rfloor} \right\rceil$$

This completes the proof. □

Theorem 4.2 *The Smarandachely scattering number of $T(C_n)$ order of $2n$ is*

$$\gamma_S(T(C_n)) = -\left\lfloor \sqrt{2n} \right\rfloor + 2 - \left\lceil \frac{2n}{\left\lfloor \sqrt{2n} \right\rfloor} \right\rceil.$$

Proof If we remove p vertices from graph $T(C_n)$, then the number of the remaining connected components is at most $\left\lfloor \frac{p}{2} \right\rfloor$. In this case the order of the largest remaining component is $m(T(C_n)-X) \geqslant \frac{2n-p}{\left\lfloor \frac{p}{2} \right\rfloor}$. So,

$$\gamma_S(T(C_n)) \quad \geqslant max_p \left\{ \frac{p}{2} - p - \frac{2n-p}{\frac{p}{2}} \right\}.$$

The function $\frac{p}{2} - p - \frac{2n-p}{\frac{p}{2}}$ takes its maximum value at $p = 2\sqrt{2n}$. Then we write this p value in the definitions of w and m to calculate the Smarandachely scattering number.

$$\gamma_S(T(C_n)) = \frac{2\left\lfloor \sqrt{2}\sqrt{n} \right\rfloor}{2} - 2\left\lfloor \sqrt{2}\sqrt{n} \right\rfloor - \frac{2n-2\left\lfloor \sqrt{2}\sqrt{n} \right\rfloor}{\frac{2\left\lfloor \sqrt{2}\sqrt{n} \right\rfloor}{2}},$$

$$\gamma_S(T(C_n)) = -2\left\lfloor \sqrt{2}\sqrt{n} \right\rfloor - \frac{2n-2\left\lfloor \sqrt{2}\sqrt{n} \right\rfloor}{\left\lfloor \sqrt{2}\sqrt{n} \right\rfloor},$$

then

$$\gamma_S(T(C_n)) = -\left\lfloor \sqrt{2n} \right\rfloor + 2 - \left\lceil \frac{2n}{\left\lfloor \sqrt{2n} \right\rfloor} \right\rceil.$$

□

Theorem 4.3 *The Smarandachely scattering number of $T(S_{1,n})$ order of $2n+1$ is*

$$r(T(S_{1,n})) = -2.$$

Proof Our proof is divided into two cases following.

Case 1 Let $|X| \leqslant \alpha(S_{1,n}) + \alpha(K_n) = 1 + (n-1) = n$ be a cut set of vertices of $T(S_{1,n})$. The number of the components in$T(S_{1,n})$is at most p, after removing p vertices. If $|X| = n$, then $w(T(S_{1,n}) - X = n$. In this case the order of the largest remaining component is

$$m(T(S_{1,n}) - X) \geqslant \left\lceil \frac{2n+1-n}{n} \right\rceil \geqslant \left\lceil \frac{n+1}{n} \right\rceil \geqslant 2.$$

Hence

$$w(T(S_{1,n}) - X) - |X| - m(T(S_{1,n}) - X) \leqslant n - n - 2 \leqslant -2.$$

Case 2 Let us take $|X| \geqslant n$. We assume $|X| = n+1$. In this case,

$$w(T(S_{1,n}) - X) \leqslant 2n + 1 - |X| = 2n + 1 - n - 1 = n,$$

$$w(T(S_{1,n}) - X) \leqslant n.$$

The order of the largest remaining component is

$$m(T(S_{1,n}) - X) \geqslant \left\lceil \frac{2n+1-|X|}{2n+1-|X|} \right\rceil = 1,$$

$$m(T(S_{1,n}) - X) \geqslant 1.$$

Hence

$$w(T(S_{1,n}) - X) - |X| - m(T(S_{1,n}) - X) \leqslant n - (n+1) - 1$$

$$w(T(S_{1,n}) - X) - |X| - m(T(S_{1,n}) - X) \leqslant -2$$

From the choice of X and the definition of the Smarandachely scattering number, we obtain $\gamma_S(T(S_{1,n})) = -2$.

It is easy to see that there is a vertex cut set X^* of $T(S_{1,n})$ such that $|\quad X^*| = n, w(T(S_{1,n}) - X^*) = n$and $m(T(S_{1,n}) - X^*) = 2$. From the definition of the Smarandachely scattering number, we have $r(T(S_{1,n})) \geqslant w(T(S_{1,n}) - X^*) - |X^*| - m(T(S_{1,n}) - X^*) = -2$. This implies that $r(T(S_{1,\,n})) = -2$. □

Theorem 4.4 *For $n \geq 3$, the Smarandachely scattering number of $T(K_2xP_n)$ of order $5n-2$ is*

$$\gamma_S(T(K_2xP_n)) = -2\left\lceil\sqrt{6+15n}\right\rceil + 8.$$

Proof There exist at most $\left\lfloor \frac{p}{4} \right\rfloor + 1$ components when p vertices are removed from the graph. The order of the largest remaining component is $m(T(K_2xP_n) - |X|) \geqslant \frac{5n-2-p}{\left\lfloor \frac{p}{4} \right\rfloor + 1}$. So,

$$\gamma_S(T(K_2xP_n)) \geqslant max_p \left\{ \frac{p}{4} + 1 - p - \frac{5n-2-p}{\frac{p}{4}+1} \right\}.$$

The function $\frac{p}{4} + 1 - p - \frac{5n-2-p}{\frac{p}{4}+1}$ takes its maximum value at $p = -4 + \frac{4}{3}\sqrt{(6+15n)}$. Then we obtain

$$\gamma_S(T(K_2xP_n)) = -2\left\lceil \sqrt{6+15n} \right\rceil + 8.$$

This completes the proof. $\square$

Theorem 4.5 *The Smarandachely scattering number of $T(K_2xC_n)$ order of $5n$ is*

$$\gamma_S\left(T\left(K_2\,x\,C_n\right)\right) \geqslant 6 - \left\lceil \sqrt{60n+24} \right\rceil.$$

Proof The number of the components is at most $\left\lfloor \frac{p+6}{4} \right\rfloor - 1$ when p vertices are removed. The number of vertices in each component is at least $m(T(K_2xC_n)$ - $|X|$ $) \geqslant \frac{5n-p}{\left\lfloor \frac{p+6}{4} \right\rfloor - 1}$. So,

$$\gamma_S(T(K_2xC_n)) \geqslant max_p \left\{ \frac{p+6}{4} - 1 - p - \frac{20n-4p}{p+2} \right\}.$$

The function $\frac{p+6}{4} - 1 - p - \frac{20n-4p}{p+2}$ takes its maximum value at $p = -2 + \frac{2}{3}\sqrt{9+3\left(20n+5\right)}$. Hence we obtain

$$\gamma_S\left(T\left(K_2\,x\,C_n\right)\right) \geqslant 6 - \left\lceil \sqrt{60n+24} \right\rceil.$$

This completes the proof. $\square$

§5. Conclusion

If we want to design a communications network, we wish it as stable as possible. Any communication network can be modeled by a connected graph. In graph theory, we have many stability measures such as connectivity, toughness, integrity and tenacity. The Smarandachely scattering number is the new parameter which measures the vulnerability of a graph G. When we design two networks which have the same number of processors, if we want to choose the more stable one from two graphs with the same number of vertices, it is enough to choose the one whose The Smarandachely scattering number is greater.

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Math.Combin.Book.Ser. Vol.4 (2009), 41-48

On the ${}_3\psi_3$ Basic Bilateral Hypergeometric Series Summation Formulas

K. R.Vasuki and G.Sharath

(Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysore-570 006, INDIA)

E-mail: vasuki_kr@hotmail.com, sharath_gns@rediffmail.com

Abstract: H.Exton has recorded two ${}_3\psi_3$ basic bilateral hypergeometric series summation formula without proof on page 305 of his book entitled *q- hypergeometric functions and applications.* In this paper, we give a proof of them.

Key Words: Basic hypergeometric series, basic bilateral hypergeometric series, contiguous functions.

AMS(2000): 33D15.

§1. Introduction

We follow the standard notation of q-series [4] and we always assume that $|q| < 1$. The q-shifted factorials $(a;q)_n$ and $(a;q)_\infty$ are defined as

$$(a;q)_n = (a)_n := \begin{cases} 1, & \text{if} \quad n = 0, \\ (1-a)(1-aq)(1-aq^2)...(1-aq^n), & \text{if} \quad n \geqslant 1 \end{cases}$$

and

$$(a;q)_\infty = (a)_\infty := (1-a)(1-aq)(1-aq^2)\cdots.$$

The basic hypergeometric series ${}_{r+1}\varphi_r$ is defined by

$${}_{r+1}\varphi_r \left[\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} ; z \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{r+1})_n}{(q)_n (b_1)_n (b_2)_n \cdots (b_r)_n} z^n, \qquad |z| < 1.$$

One of the most classical identities in q-series is the q-binomial theorem, due to Cauchy:

[1]Received Oct.3, 2009. Accepted Nov. 8, 2009.

$$ {}_1\varphi_0\left[\begin{matrix} a \\ - \end{matrix};z\right] = \frac{(az)_\infty}{(z)_\infty}, \qquad |z|<1, \tag{1.1}$$

Another classical q-series identity in q-series is Heine's q-analogue of the Gauss ${}_2F_1$ summation formula:

$$ {}_2\varphi_1\left[\begin{matrix} a,b \\ c \end{matrix};\frac{c}{ab}\right] = \frac{(c/a)_\infty(c/b)_\infty}{(c)_\infty(c/ab)_\infty}, \qquad \left|\frac{c}{ab}\right|<1. \tag{1.2}$$

Heine deduced (1.2) as a particular case of his transformation formula [5]

$$ {}_2\varphi_1\left[\begin{matrix} a,b \\ c \end{matrix};z\right] = \frac{(b)_\infty(az)_\infty}{(c)_\infty(z)_\infty}\ {}_2\varphi_1\left[\begin{matrix} c/b,z \\ az \end{matrix};b\right], |z|<1, |b|<1. \tag{1.3}$$

Another interesting transformation formula due to Sear's [7] is

$$ {}_3\varphi_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix};\frac{de}{abc}\right] = \frac{(e/a)_\infty(de/bc)_\infty}{(e)_\infty(de/abc)_\infty}\ {}_3\varphi_2\left[\begin{matrix} a,d/b,d/c \\ d,de/bc \end{matrix};e/a\right], \tag{1.4}$$

$|de/abc|<1$, $|e/a|<1$. The basic bilateral hypergeometric series ${}_r\psi_r$is defined by

$$ {}_r\psi_r\left[\begin{matrix} a_1,a_2,\cdots,a_r \\ b_1,b_2,\cdots,b_r \end{matrix};z\right] := \sum_{n=-\infty}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n(b_2)_n\cdots(b_r)_n}z^n,$$

$\left|\frac{b_1b_2\cdots b_r}{a_1a_2\cdots a_r}\right|<|z|<1$. There are many generalizations of q-binomial theorem (1.1) of which, one of the interesting is the following Ramanujan's ${}_1\psi_1$ summation [1] [6]:

$$ {}_1\psi_1\left[\begin{matrix} a \\ b \end{matrix};z\right] = \frac{(az)_\infty(b/a)_\infty(q/az)_\infty(q)_\infty}{(z)_\infty(q/a)_\infty(b/az)_\infty(b)_\infty}, \quad |b/a|<|z|<1. \tag{1.5}$$

A variety of proofs have been given of (1.5). For more details of (1.5), one may refer [1], [4]. H. Exton [3, p. 305] has given following two ${}_3\psi_3$ basic bilateral series summation formula without proof :

$$ {}_3\psi_3\left[\begin{matrix} a,b,cq \\ d,bq,c \end{matrix};\frac{1}{a}\right] = \frac{(1-(b/c))(d/b)_\infty(bq/a)_\infty(q)_\infty^2}{(1-(1/c))(q/b)_\infty(q/a)_\infty(bq)_\infty(d)_\infty}, \tag{1.6}$$

$|d|<1$, $|1/a|<1$ and

$$ {}_3\psi_3\left[\begin{matrix} a,b,cq \\ d,bq,c \end{matrix};\frac{q}{a}\right] = \frac{(1-(c/b))(d/b)_\infty(bq/a)_\infty(q)_\infty^2}{(1-c)(q/b)_\infty(q/a)_\infty(bq)_\infty(d)_\infty}, \tag{1.7}$$

$|d/q| < 1$, $|q/a| < 1$. Exton [3, p. 305] has incorrectly given $(q/c)_\infty$ instead of $(q/b)_\infty$ in the denominator of (1.7). W. Chu [2], deduced (1.6) and (1.7) as a special cases of his integral-summation formula. In this paper, we give a proof of (1.6) and (1.7) on the lines of G. E. Andrews and R. Askey [1] proof of (1.5).

§2. Proof of (1.6) and (1.7)

Lemma 2.1 *We have*

$$\frac{a}{d}(1-d)\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f\end{matrix};z\right] + (1-(a/d))\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ dq,e,f\end{matrix};z\right]$$

$$= \frac{(1-d)(1-(e/q))(1-(f/q))}{z(1-(b/q))(1-(c/q))}\ {}_3\psi_3\left[\begin{matrix} a,b/q,c/q \\ d,e/q,f/q\end{matrix};z\right], \tag{2.1}$$

$$\frac{((d/q)-b)}{1-b}\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f\end{matrix};z\right] - \frac{(d-a)}{(q-a)}\ {}_3\psi_3\left[\begin{matrix} a/q,bq,c \\ d,e,f\end{matrix};z\right]$$

$$= \frac{z((a/q)-b)(1-c)}{(1-e)(1-f)}\ {}_3\psi_3\left[\begin{matrix} a,bq,cq \\ d,eq,fq\end{matrix};z\right], \tag{2.2}$$

$$\frac{b(d-1)(d-a)}{d(d-b)}\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f\end{matrix};z\right] = (1-(a/d))\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ dq,e,f\end{matrix};z\right]$$

$$+\frac{(d-1)(d-a)(1-(e/q))(1-(f/q))}{z((d/q)-(b/q))(1-(c/q))(q-a)}\ {}_3\psi_3\left[\begin{matrix} a/q,b,c/q \\ d,e/q,f/q\end{matrix};z\right], \tag{2.3}$$

$$\left[f-(a/q)-\frac{(d-a)}{(q-a)}\right]\ {}_3\psi_3\left[\begin{matrix} a/q,bq,c \\ d,e,f\end{matrix};z\right] = \frac{((d/q)-f)(f-(a/q))}{(1-f)}$$

$${}_3\psi_3\left[\begin{matrix} a/q,bq,c \\ d,e,fq\end{matrix};zq\right] - \frac{((d/q)-f)}{(1-f)}\ {}_3\psi_3\left[\begin{matrix} a,bq,c \\ d,e,fq\end{matrix};z\right]. \tag{2.4}$$

Proof of (2.1). It is easy to see that

$$a\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ dq,e,f\end{matrix};zq\right] + \frac{a(1-d)}{d}\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f\end{matrix};z\right]$$

$$= \frac{a}{d}\sum_{n=-\infty}^{\infty}\frac{(a)_n(b)_n(c)_n}{(dq)_{n-1}(e)_n(f)_n}z^n\left[\frac{dq^n}{(1-dq^n)}+1\right]$$

$$= \frac{a}{d}\sum_{n=-\infty}^{\infty}\frac{(a)_n(b)_n(c)_n}{(dq)_n(e)_n(f)_n}z^n.$$

Hence,

$$a\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ dq,e,f \end{matrix};zq\right]+\frac{a(1-d)}{d}\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f \end{matrix};z\right]=\frac{a}{d}\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ dq,e,f \end{matrix};z\right]. \tag{2.5}$$

Also, we have

$${}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f \end{matrix};z\right]-a\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f \end{matrix};zq\right]=\sum_{n=-\infty}^{\infty}\frac{(a)_{n+1}(b)_n(c)_n}{(d)_n(e)_n(f)_n}z^n$$

$$=\frac{(1-(d/q))(1-(e/q))(1-(f/q))}{z(1-(b/q))(1-(c/q))}\sum_{n=-\infty}^{\infty}\frac{(a)_n(b/q)_n(c/q)_n}{(d/q)_n(e/q)_n(f/q)_n}z^n.$$

Thus,

$${}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f \end{matrix};z\right]-a\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f \end{matrix};zq\right]$$

$$=\frac{(1-(d/q))(1-(e/q))(1-(f/q))}{z(1-(b/q))(1-(c/q))}\ {}_3\psi_3\left[\begin{matrix} a,b/q,c/q \\ d/q,e/q,f/q \end{matrix};z\right]. \tag{2.6}$$

Changing d to dq in (2.6) and then adding resulting identity with (2.5), we obtain (2.1).

Proof of (2.2). We have

$$\frac{((d/q)-b)}{(1-b)}\sum_{n=-\infty}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n(f)_n}z^n-\frac{(d-a)}{(q-a)}\sum_{n=-\infty}^{\infty}\frac{(q/a)_n(bq)_n(c)_n}{(d)_n(e)_n(f)_n}z^n$$

$$=\sum_{n=-\infty}^{\infty}\frac{(a)_{n-1}(bq)_{n-1}(c)_n}{(d)_n(e)_n(f)_n}z^n\left[((d/q)-b)(1-aq^{n-1})-((d/q)-(a/q))(1-bq^n)\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{(a)_{n-1}(bq)_{n-1}(c)_n}{(d)_n(e)_n(f)_n}z^n((a/q)-b)(1-dq^{n-1})$$

$$=\frac{z((a/q)-b)(1-c)}{(1-e)(1-f)}\sum_{n=-\infty}^{\infty}\frac{(a)_n(bq)_n(cq)_n}{(d)_n(eq)_n(fq)_n}z^n.$$

This proves (2.2).

Proof of (2.3). Changing b to b/q, c to c/q, e to e/q and f to f/q in (2.2), and multiplying throughout by $\frac{q(1-d)(1-(e/q))(1-(f/q))}{z(1-(c/q))(d-b)}$ and adding the resulting identity with (2.1), we find (2.3).

Proof of (2.4). From [8], we have

$$(1-f)\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f \end{matrix};z\right] - ((d/q)-f)\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,fq \end{matrix};zq\right]$$

$$= \frac{z(1-a)(1-b)(1-c)}{(1-fq)(1-e)}\ {}_3\psi_3\left[\begin{matrix} aq,bq,cq \\ d,eq,fq^2 \end{matrix};z\right],$$

and

$$\frac{((d/q)-a)}{(1-a)}\ {}_3\psi_3\left[\begin{matrix} a,b,c \\ d,e,f \end{matrix};z\right] - \frac{((d/q)-f)}{(1-f)}\ {}_3\psi_3\left[\begin{matrix} aq,b,c \\ d,e,fq \end{matrix};z\right].$$

$$= \frac{z(f-a)(1-b)(1-c)}{(1-f)(1-fq)(1-e)}\ {}_3\psi_3\left[\begin{matrix} aq,bq,cq \\ d,eq,fq^2 \end{matrix};z\right]$$

Eliminating ${}_3\psi_3\left[\begin{matrix} aq,bq,cq \\ d,eq,fq \end{matrix};z\right]$ between above two identities and then replacing a by a/q and b by bq , we obtain(2.4).

Proof of (1.6). Setting $c=cq,\ e=bq,\ f=c$ and $z=1/a$ in (2.4), we deduce that

$$\left[c-(a/q)-\frac{(d-a)}{(q-a)}\right]\ {}_2\psi_2\left[\begin{matrix} a/q,cq \\ d,c \end{matrix};1/a\right] = \frac{((d/q)-c)(c-(a/q))}{(1-c)}\ {}_1\psi_1\left[\begin{matrix} a/q \\ a \end{matrix};q/a\right]$$

$$-\frac{((d/q)-c)}{(1-c)}\ {}_1\psi_1\left[\begin{matrix} a \\ d \end{matrix};1/a\right].$$

Employing (1.5) in the right side of the above, we obtain

$$ {}_2\psi_2\left[\begin{matrix} a/q,cq \\ d,c \end{matrix};1/a\right] = 0. \tag{2.7}$$

Let

$$f(d) = {}_3\psi_3\left[\begin{matrix} a,b,cq \\ d,bq,c \end{matrix};1/a\right].$$

As a function of d, $f(d)$ is clearly analytic for $|d|<1$ and $|a|>1$. Setting $c=cq$, $e=bq$, $f=c$ and $z=1/a$ in (2.3) and then employing (2.7), we find that

$$f(d)=\frac{(1-(d/b))}{(1-d)}f(dq). \tag{2.8}$$

Iterating (2.8) $n-1$ times , we find that

$$f(d)=\frac{(d/b)_n}{(d)_n}f(dq^n).$$

Since $f(d)$ is analytic for $|d|<1$, $|a|>1$, by letting $n\to\infty$, we obtain

$$f(d)=\frac{(d/b)_\infty}{(d)_\infty}f(0).$$

Setting $c=cq$, $d=c$, $e=bq$ in (1.4), we deduce that

$$
\begin{aligned}
&{}_3\varphi_2\left[\begin{matrix} a,b,cq \\ bq,c\end{matrix};1/a\right]\\
&=\frac{(bq/a)_\infty(q)_\infty}{(bq)_\infty(1/a)_\infty}\sum_{n=0}^{\infty}\frac{(a)_n(c/b)_n(1/q)_n}{(q)_n(c)_n(q)_{n-1}}(bq/a)^n\\
&=\frac{(bq/a)_\infty(q)_\infty}{(bq)_\infty(1/a)_\infty}\frac{(1-a)(1-(c/b))(1-(1/q))(bq/a)}{(1-q)(1-c)}\sum_{n=0}^{\infty}\frac{(aq)_n(cq/b)_n(1)_n}{(q)_n(cq)_n(q^2)_n}(bq/a)^n\\
&=\frac{b(1-(c/b))(bq/a)_\infty(q)_\infty}{(1-c)(bq)_\infty(q/a)_\infty}.
\end{aligned}
$$

Thus,

$$f(q)=\frac{(1-(b/c))(bq/a)_\infty(q)_\infty}{(1-(1/c))(bq)_\infty(q/a)_\infty}.$$

Setting $d=q$ in (2.9), and using the above, we find that

$$f(0)=\frac{(1-(b/c))(bq/a)_\infty(q)_\infty^2}{(1-(1/c))(bq)_\infty(q/a)_\infty(q/b)_\infty}.$$

Using this in (2.9), we deduce that

$$f(d)=\frac{(1-(b/c))(bq/a)_\infty(d/b)_\infty(q)_\infty^2}{(1-(1/c))(bq)_\infty(q/a)_\infty(q/b)_\infty(d)_\infty}.$$

This completes the proof of (1.6).

Proof of (1.7). Setting $c = cq,\ e = bq,\ f = c$ and $z = q/a$ in (2.4) and then employing (1.5), we find that

$$ {}_2\psi_2\left[\begin{matrix} a/q, cq \\ d, c \end{matrix} ; q/a\right] = 0. \tag{2.9}$$

Let

$$ f(d) := {}_3\psi_3\left[\begin{matrix} a, b, cq \\ d, bq, c \end{matrix} ; q/a\right].$$

As a function of d, $f(d)$ is clearly analytic for $|d| < 1$, when $|q/a| < 1$. Setting $c = cq$, $e = bq$, $f = c$ and $z = q/a$ in (2.3) and then employing (2.10), we find that

$$ f(d) = \frac{(1-(d/b))}{(1-d)} f(dq).$$

Iterating the above $n-1$ times , we get

$$ f(d) = \frac{(d/b)_n}{(d)_n} f(dq^n).$$

Since $f(d)$ is analytic for $|d| < 1$, $|q/a| < 1$, by letting $n \to \infty$, we obtain

$$ f(d) = \frac{(d/b)_\infty}{(d)_\infty} f(0).$$

Setting $c = bq$, $z = q^2/a$ in (1.3) and employing (1.1), we obtain

$$ {}_2\varphi_1\left[\begin{matrix} a, b \\ bq \end{matrix} ; q^2/a\right] = \frac{(bq/a)_\infty (q)_\infty}{b(bq)_\infty (q/a)_\infty}.$$

Also by (1.2), we deduce that

$$ \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(q)_n (bq)_n} (q/a)^n = \frac{(bq/a)_\infty (q)_\infty}{(bq)_\infty (q/a)_\infty}.$$

Thus,

$$f(q) ={}_3\varphi_2\left[\begin{matrix} a,b,cq \\ bq,c \end{matrix};q/a\right]$$
$$=\frac{1}{(1-c)}\left[{}_2\varphi_1\left[\begin{matrix} a,b \\ bq \end{matrix};q/a\right]-c\ {}_2\varphi_1\left[\begin{matrix} a,b \\ bq \end{matrix};q^2/a\right]\right]$$
$$=\frac{(1-(c/b))(bq/a)_\infty(q)_\infty}{(1-c)(q/a)_\infty(bq)_\infty}.$$

Now setting in $d=q$ in (2.11) and employing above, we find that

$$f(0)=\frac{(1-(c/b))(bq/a)_\infty(q)_\infty^2}{(1-c)(q/a)_\infty(bq)_\infty(q/b)_\infty}.$$

Using this in (2.11), we deduce (1.7).

Acknowledgement

Authors are thankful to DST, New Delhi for awarding research project [No. SR/S4/MS:517/08] under which this work has been done.

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$$1+\frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)}x+\frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})}x^2+\cdots,$$

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Math.Combin.Book.Ser. Vol.4 (2009), 49-55

Minimal Retraction of Space-time and Their Foldings

A. E. El-Ahmady and H. Rafat

(Department of Mathematics, Faculty Science of Tanta University, Tanta, Egypt)

E-mail:

Abstract: A *Smarandache multi-spacetime* is such a union spacetime $\bigcup_{i=1}^{n} S_i$ of spacetimes $S_1, S_2, \cdots, S_n$ for an integer $n \geq 1$. In this article, we will be deduced the geodesics of space-time, i.e., a Smarandache multi-spacetime with $n = 1$ by using Lagrangian equations. The deformation retract of space-time onto itself and into a geodesics will be achieved. The concept of retraction and folding of zero dimension space-time will be obtained.The relation between limit of folding and retraction presented.

Key Words: Folding, deformation retract, space-time, Smarandache multi-spacetime.

AMS(2000): 53A35, 51H05, 58C05, 51F10, 58B34.

§1. Introduction

The folding of a manifold was, firstly introduced by Robertson in [1977] [14]. Since then many authers have studied the folding of manifolds such as in [4,6,12,13]. The deformation retracts of the manifolds defined and discussed in [5,7]. In this paper, we will discuss the folding restricted by a minimal retract and geodesic. We may also mention that folding has many important technical applications, for instance, in the engineering problems of buckling and post-buckling of elastic and elastoplastic shells [1]. More studies and applications are discussed in [4], [8], [9], [10], [13].

§2. Definitions

1. A subset A of a topological space X is called a retract of X, if there exists a continuous map $r : X \to A$ such that ([2]):

(i) X is open;
(ii)$r(a) = a, \ \forall a \in A$.

2. A subset A of atopological space X is said to be a deformation retract if there exists a retraction $r : X \to A$, and a homotopy $f : X \times I \to X$ such that([2]):

$f(x, 0) = x, \forall, x \in X$;

[1]Received Oct.15, 2009. Accepted Nov. 18, 2009.

$f(x,1) = r(x), \forall x \in X;$
$f(a,t) = a, \forall a \in A, t \in [0,1].$

3. Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : J \to M$,the induced path $f \circ \gamma : J \to N$ is a piecewise geodesic and of the same length as γ ([14]). If $f$does not preserve the lengths, it is called topological folding.

4. Let M be an m-dimensional manifold. M is said to be minimal m-dimensional manifold if the mean curvature vanishes everywhere, i.e., $H(\sigma.p) = 0$ for all $p \in M$ ([3]).

5. A subset A of a minimal manifold M is a minimal retraction of M, if there exists a continuous map $r : M \to A$such that ([12]):

(i)M is open;
(ii)$r(M) = A$;
(iii)$r(a) = a, \forall a \in A$;
(iv)$r(M)$is minimal manifold.

§3. Main Results

Using the Neugebaure-Bcklund transformation, the space-time T take the form [11]

$$ds^2 = d\,t^2 - d\,p^2 - d\,z^2 - p^2 d\,\phi^2 \tag{1}$$

Using the relationship between the cylindrical and spherical coordinates, the metric becomes

$$\begin{aligned}\overline{ds}^2 &= r^2(\sin^2\theta_2 - \cos^2\theta_2)\,\overline{d\theta}_2^2 - r^2\sin^2\theta_2\,\overline{d\theta}_1^2 + (\cos^2\theta_2 - \sin^2\theta_2)\,\overline{dr}^2 \\ &\quad - r^2\sin^2\theta_1\sin^2\theta_2\,\overline{d\varphi}^2 - 4r\sin\theta_2\cos\theta_2 d\theta_2 dr.\end{aligned}$$

The coordinates of space-time T are:

$$\left.\begin{aligned} y_1 &= \sqrt{c_1(r,\theta_2) - r^2\sin^2\theta_2\theta_1^2} \\ y_2 &= \sqrt{4r^2\cos 2\theta_2 + k_1} \\ y_3 &= \sqrt{r^2\cos 2\theta_2 + c_3(\theta_2)} \\ y_4 &= \sqrt{c_4(r,\theta_1,\theta_2) - r^2\sin^2\theta_1\sin^2\theta_2\phi^2} \end{aligned}\right\} \tag{2}$$

where c_1, k_1, c_3, c_4 are the constant of integrations. Applying the transformation

$x_1^2 = y_1^2 - c_1(r,\theta_2),$
$x_2^2 = y_2^2 - k_1,$
$x_3^2 = y_3^2 - c_3(\theta_2),$
$x_4^2 = y_4^2 - c_4(r,\theta_1,\theta_2)$

Then, the coordinates of space-time T becomes:

$$\left.\begin{array}{l} x_1 = ir\sin\theta_2\theta_1 \\ x_2 = 2r\sqrt{\cos 2\theta_2} \\ x_3 = r\sqrt{\cos 2\theta_2} \\ x_4 = ir\ \sin\theta_1\sin\theta_2\phi. \end{array}\right\} \tag{3}$$

Now, we apply Lagrangian equations

$$\frac{d}{ds}\left(\frac{\partial T}{\partial G_i}\right) - \frac{\partial T}{\partial G_i} = 0, i = 1, 2, 3, 4.$$

to find a geodesic which is a subset of the space-time T . Since

$$\begin{aligned} T &= \frac{1}{2}\{-r^2\cos 2\theta_2\theta_2'^2 - r^2\sin^2\theta_2\theta_1'^2 + \cos 2\theta_2 r'^2 - r^2\sin^2\theta_1\sin^2\theta_2\phi'^2 \\ &\quad -2r\sin 2\theta_2\theta_2' r'\} \end{aligned}$$

then, the Lagrangian equations for space-time T are:

$$\frac{d}{ds}(r^2\sin^2\theta_2\theta_1') + (r^2\sin\theta_1\cos\theta_1\sin^2\theta_2\phi^{'2}) = 0 \tag{4}$$

$$\begin{aligned} &\tfrac{d}{ds}(r^2\cos 2\theta_2\theta_2' + r\sin\theta_2 r') + (r^2\sin 2\theta_2\theta_2'^2 + r^2\sin\theta_2\cos\theta_2\theta_1'^1 \\ &+\sin 2\theta_2 r'^2 + r^2\sin^2\theta_1\sin\theta_2\cos\theta_2\phi'^2 + 2r\cos 2\theta_2\theta_2' r') = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} &\tfrac{d}{ds}(\cos 2\theta_2 r' - r\sin 2\theta_2\theta_2') + (r\cos 2\theta_2\theta_2'^2 + r\sin^2\theta_2\theta_1'^2 + \\ &r\sin^2\theta_1\sin^2\theta_2\phi'^2 + \sin 2\theta_2\theta_2' r') = 0 \end{aligned} \tag{6}$$

$$\frac{d}{ds}(r^2\sin^2\theta_1\sin^2\theta_2\phi') = 0. \tag{7}$$

From equation (7) we obtain $r^2\sin^2\theta_1\sin^2\theta_2\phi^1 =$ constant μ. If $\mu = 0$, we obtain the following cases:

(i) If $r = 0$, hence we get the coordinates of space-time T_1, which are defined as

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0,$$

which is a hypersphere T_1, $x_1^2 - x_2^2 - x_3^2 - x_4^2 = 0$ on the null cone since the distance between any two different points equal zero, it is a minimal retraction and geodesic.

(ii) If $sin^2\theta_1 = 0$, we get

$$x_1 = 0, x_2 = 2r\sqrt{\cos 2\theta_2}, x_3 = r\sqrt{\cos 2\theta_2}, x_4 = 0.$$

Thus, $x_1^2+x_2^2+x_3^2+x_4^2$=$5r^2\cos 2\theta_2$, which is a hypersphere S_1 in space-time T with $x_1 = x_4 = 0$. It is a geodesic and retraction.

(iii) If $\sin^2\theta_2 = 0$, then $\theta_2 = 0$ we obtain the following geodesic retraction

$$x_1 = 0, x_2 = 2r, x_3 = r, x_4 = 0, \quad x_1^2 + x_2^2 + x_3^2 - x_1^2 = 5r^2,$$

which is the hypersphere $S_2 \subset T$ with $x_1 = x_4 = 0$.

(iv) If $\phi' = 0$ this yields the coordinate of $T_2 \subset T$ given by

$$x_1 = ir \sin\theta_2\theta_1, x_2 = 2r\sqrt{\cos 2\theta_2}, x_3 = r\sqrt{\cos 2\theta_2}, x_4 = 0.$$

It is worth nothing that $x_4 = 0$ is a hypersurface $T_2 \subset T$. Hence, we can formulate the following theorem.

Theorem 1 *The retractions of space-time is null geodesic, geodesic hyperspher and hypersurface.*

Lemma 1 *In space-time the minimal retraction induces null-geodesic.*

Lemma 2 *A minimal geodesic in space-time is a necessary condition for minimal retration.*

The deformation retract of the space-time T is defined as

$$\rho : T \times I \to T$$

where T is the space-time and I is the closed interval [0,1]. The retraction of the space-time T is defined as

$$R : T \to T_1, T_2, S_1 \text{ and } S_2.$$

The deformation retract of space-time T into a geodesic $T_1 \subset T$ is defined by

$$\begin{aligned}\rho(m,t) &= (1-t)\{ir\ \sin\theta_2\,\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2},\\ &\quad ir \sin\theta_1 \sin\theta_2\phi\} + t\{0,0,0,0\}.\end{aligned}$$

where $\rho(m,0) = \{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, ir\sin\theta_1\sin\theta_2\phi\}$, $\rho(m,1) = \{0,0,0,0\}$.

The deformation retract of space-time T into a geodesic $T_2 \subset T$ is defined as

$$\begin{aligned}\rho(m,t) &= (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, ir\sin\theta_1\sin\theta_2\phi\}\\ &+ t\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, 0\}.\end{aligned}$$

The deformation retract of space-time T into a geodesic $S_1 \subset T$ is defined by

$$\begin{aligned}\rho(m,t) &= (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, ir\sin\theta_1\sin\theta_2\phi\}\\ &+ t\{0, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, 0\}.\end{aligned}$$

The deformation retract of space-time T into a geodesic $S_2 \subset T$ is defined as

$$\rho(m,t) = (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, ir\sin\theta_1\sin\theta_2\phi\} + t\{0, 2r, r, 0\}.$$

Now we are going to discuss the folding $\Im$ of the space-time T. Let $\Im : T \to T$, where

$$\Im(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, |x_4|) \tag{8}$$

An isometric folding of the space-time T into itself may be defined as

$$\begin{aligned} \Im : \quad & \{ir\sin\theta_2\theta_1,\ 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, ir\sin\theta_1\sin\theta_2\phi\} \\ \to \quad & \{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\}. \end{aligned}$$

The deformation retract of the folded space-time T into the folded geodesic T_1 is

$$\begin{aligned} \rho_\Im : \quad & \{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\} \times I \\ \to \quad & \{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\} \end{aligned}$$

with

$$\rho_\Im(m,t) = (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\} + t\{0,0,0,0\}.$$

The deformation retract of the folded space-time T into the folded geodesicT_2 is

$$\begin{aligned} \rho_\Im(m,t) &= (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\} \\ &+ t\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, 0\}. \end{aligned}$$

The deformation retract of the folded space-time T into the folded geodesic S_1 is

$$\begin{aligned} \rho_\Im(m,t) &= (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\} \\ &+ t\{0, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, 0\}. \end{aligned}$$

The deformation retract of the folded space-time T into the folded geodesic S_2 is

$$\begin{aligned} \rho_\Im(m,t) &= (1-t)\{ir\sin\theta_2\theta_1, 2r\sqrt{\cos 2\theta_2}, r\sqrt{\cos 2\theta_2}, |ir\sin\theta_1\sin\theta_2\phi|\} \\ &+ t\{0, 2r, r, 0\} \end{aligned}$$

Then, the following theorem has been proved.

Theorem 2 *Under the defined folding, the deformation retract of the folded space-time into the folded geodesics is the same as the deformation retract of space-time into the geodesics.*

Now, let the folding be defined as:

$$\Im^*(x_1, x_2, x_3, x_4) = (x, |x_2|, x_3, x_4). \tag{9}$$

The isometric folded space-time $\Im(T)$ is

$$\bar{R} = \{ir\sin\theta_2\theta_1, \left|2r\sqrt{\cos 2\theta_2}\right|, r\sqrt{\cos 2\theta_2}, ir\sin\theta_1\sin\theta_2\phi\}.$$

Hence, we can formulate the following theorem.

Theorem 3 *The deformation retract of the folded space-time ,i.e., $\rho\Im^*(T)$ is different from the deformation retract of space-time under condition (9).*

Now let $\Im_1 : T^n \to T^n$,

$\Im_2 : \Im_1(T^n) \to \Im_1(T^n)$,

$\Im_3 : \Im_2(\Im_1(T^n)) \to \Im_2(\Im_1(T^n)), \cdots$,

$\Im_n : \Im_{n-1}(\Im_{n-2}...(\Im_1(T^n))...)) \to \Im_{n-1}(\Im_{n-2}...(\Im_1(T^n))...))$,

$\lim\limits_{n\to\infty} \Im_{n-1}(\Im_{n-2}...(\Im_1(T^n))...)) = n-1$ dimensional space-time T^{n-1}.

Let $h_1 : T^{n-1} \to T^{n-1}$,

$h_2 : h_1(T^{n-1}) \to h_1(T^{n-1})$,

$h_3 : h_2(h_1(T^{n-1})) \to h_2(h_1(T^{n-1}), ...,$

$h_m : h_{m-1}(h_{m-2}...(h_1(T^{n-1}))...)) \to h_{m-1}(h_{m-2}...(h_1(T^{n-1}))...))$,

$\lim h_m(h_m : h_{m-1}(h_{m-2}...(h_1(T^{n-1}))...)) = n-2$ dimensional space-time T^{n-2}.

Consequently, $\lim\limits_{s\to\infty}\lim\limits_{m\to\infty}\lim\limits_{n\to\infty}...k_s(h_m(\Im_n(T^n))) =$ 0-dimensional space-time. Hence, we can formulate the following theorem.

Theorem 4 *The end of the limits of the folding of space-time T^n is a 0-dimensional geodesic, it is a minimal retraction.*

Now let f_1 be the foldings and r_i be the retractions. then we have

$T^n \xrightarrow{f_1^1} T_1^n \xrightarrow{f_2^1} T_2^n \longrightarrow \cdots T_{n-1}^n \xrightarrow{\lim f_i^1} T^{n-1}$,

$T^n \xrightarrow{r_1^1} T_1^n \xrightarrow{r_2^1} T_2^n \longrightarrow \cdots T_{n-1}^n \xrightarrow{\lim r_i^1} T^{n-1}$,

$T^n \xrightarrow{f_1^2} T_1^{n-1} \xrightarrow{f_2^2} T_2^{n-1} \longrightarrow \cdots T_{n-1}^n \xrightarrow{\lim f_i^2} T^{n-2}, \cdots$,

$T^{n-1} \xrightarrow{r_1^1} T_1^{n-1} \xrightarrow{r_2^2} T_2^{n-1} \longrightarrow \cdots\ T_{n-1}^n \xrightarrow{\lim r_i^2} T^{n-2}, \cdots$,

$T^1 \xrightarrow{f_1^n} T_1^1 \xrightarrow{f_2^n} T_2^1 \longrightarrow \cdots T_{n-1}^1 \xrightarrow{\lim f_i^n} T^0$,

$T^1 \xrightarrow{r_1^n} T_1^1 \xrightarrow{r_2^n} T_2^1 \longrightarrow \cdots T_{n-1}^1 \xrightarrow{\lim f_i^n} T^0$.

Then the end of the limits of foldings = the limit of retractions = 0-dimensional space-time. Whence, the following theorem has been proved.

Theorem 5 *In space-time the end of the limits of foldings of T^n into itself coincides with the minimal retraction.*

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Math.Combin.Book.Ser. Vol.4 (2009), 56-62

Efficient Domination in Bi-Cayley Graphs

T. Tamizh Chelvam and Sivagnanam Mutharasu

(Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli-627012, India)

E-mail: tamche59@gmail.com, skannanmunna@yahoo.com

Abstract: A Cayley graph is constructed out of a group Γ and its generating set X and it is denoted by $\mathbb{C}(\Gamma, X)$. A *Smarandachely n-Cayley graph* is defined to be $G = ZC(\Gamma, X)$, where $V(G) = \Gamma \times \mathbb{Z}_n$ and $E(G) = \{((x,0),(y,1))_a, ((x,1),(y,2))_a, \cdots, ((x,n-2),(y,n-1))_a : x, y \in \Gamma, a \in X$ such that $y = x * a\}$. Particularly, a Smarandachely 2-Cayley graph is called as a *Bi-Cayley graph*, denoted by $B\mathbb{C}(\Gamma, X)$. Necessary and sufficient conditions for the existence of an efficient dominating set and an efficient open dominating set in Bi-Cayley graphs are determined.

Key Words: Cayley graphs, Smarandachely n-Cayley graph, Bi-Cayley graphs, efficient domination, efficient open domination, covering of a graph.

AMS(2000): 05C69.

§1. Introduction

The terminology and notation in this paper follows that found in [3]. The fact that Cayley graphs are excellent models for interconnection networks, investigated in connection with parallel processing and distributed computation. The concept of domination for Cayley graphs has been studied by various authors and one can refer to [2, 4, 6]. I.J. Dejter, O. Serra [2], J.Huang, J-M. Xu [4] obtained some results on efficient dominating sets for Cayley graphs. The existence of independent perfect dominating sets in Cayley graphs was studied by J.Lee [6]. Tamizh Chelvam and Rani [8-10], obtained the domination, independent domination, total domination and connected domination numbers for some Cayley graphs constructed on $\mathbb{Z}_n$ for some generating set of $\mathbb{Z}_n$.

Let $(\Gamma, *)$ be a group with e as the identity and X be a symmetric generating set(if $a \in X$, then $a^{-1} \in X$) with $e \notin X$. The Cayley graph $G = \mathbb{C}(\Gamma, X)$, where $V(G) = \Gamma$ and $E(G) = \{(x,y)_a / x, y \in V(G), a \in X$ such that $y = x * a\}$. Since X is a generating set for Γ, $\mathbb{C}(\Gamma, X)$ is a connected and regular graph of degree $|X|$. The Bi-Cayley graph is defined as $G = B\mathbb{C}(\Gamma, X)$, where $V(G) = \Gamma \times \{0, 1\}$ and $E(G) = \{((x,0),(y,1))_a / x, y \in \Gamma, a \in X$ such that $y = x * a\}$. Now the operation $+$ is defined by $(x,0) + (y,1) = (x * y, 1)$ and $(x,0) + (y,0) = (x * y, 0)$.

[1]Received Oct.9, 2009. Accepted Nov. 20, 2009.

The *Smarandachely n-Cayley graph* is defined to be $G = ZC(\Gamma, X)$, where $V(G) = \Gamma \times \mathbb{Z}_n$ and $E(G) = \{((x,0),(y,1))_a, ((x,1),(y,2))_a, \cdots, ((x,n-2),(y,n-1))_a : x, y \in \Gamma, a \in X$ such that $y = x * a\}$. When $n = 2$, the Smarandachely n-Cayley graphs are called as *Bi-Cayley graphs*. By the definition of Bi-Cayley graph, it is a regular graph of degree $|X|$.

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V - S$ is adjacent to an element u of S. The domination number $\gamma(G)$ is the minimum cardinality among all the dominating sets in G [3] and a corresponding dominating set is called a γ-set. A dominating set S is called an efficient dominating set if for every vertex $v \in V, |N[v] \cap S| = 1$. Note that if S is an efficient dominating set then $\{N[v] : v \in S\}$ is a partition of $V(G)$ and if G has an efficient dominating set, then all efficient dominating sets in G have the same cardinality namely $\gamma(G)$. A set $S \subseteq V$ is called a total dominating set if every vertex $v \in V$ is adjacent to an element $u(\neq v)$ of S. The total domination number $\gamma_t(G)$ of G equals the minimum cardinality among all the total dominating sets in G [3] and a corresponding total dominating set is called a γ_t-set. A dominating set S is called an efficient open dominating set if for every vertex $v \in V,\ |N(v) \cap S| = 1$.

A graph $\tilde{G}$ is called covering of G with projection $f : \tilde{G} \to G$ if there is a surjection $f : V(\tilde{G}) \to V(G)$ such that $f|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(G)$ and $\tilde{v} \in f^{-1}(v)$. Also the projection $f : \tilde{G} \to G$ is said to be an *n-fold covering* if f is $n-$to-one.

In this paper, we prove that the Bi-Cayley graph obtained from Cayley graph for an Abelian group $(\Gamma, *)$ has an efficient dominating set if and only if it is a covering of the graph $\overline{K_n \times K_2}$. It is also proved that the Bi-Cayley graph obtained from Cayley graph for an Abelian group $(\Gamma, *)$ has an efficient open dominating set if and only if it is a covering of the graph $K_{n,n}$.

Theorem 1.1([4]) *Let G be a $k-$regular graph. Then $\gamma(G) \geq \frac{|V(G)|}{k+1}$, with the inequality if and only if G has an efficient dominating set.*

Theorem 1.2([6]) *Let $p : \tilde{G} \to G$ be a covering and let S be a perfect dominating set of G. The $p^{-1}(S)$ is a perfect dominating set of $\tilde{G}$. Moreover, if S is independent, then $p^{-1}(S)$ is independent.*

Theorem 1.3([3]) *If G has an efficient open dominating set S, then $|S| = \gamma_t(G)$ and all efficient open dominating sets have the same cardinality.*

§2. Efficient Domination and Bi-Cayley Graphs

In this section, we find the necessary and sufficient condition for the existence of an efficient dominating set in $B\mathbb{C}(\Gamma, X)$. Since $B\mathbb{C}(\Gamma, X)$ is regular bi-partite graph, in $B\mathbb{C}(\Gamma, X)$ every efficient dominating set S is of the form $S = A \cup B$ where $A \subseteq (\Gamma \times 0) \cap S$ and $B \subseteq (\Gamma \times 1) \cap S$ with $|A| = |B| = \frac{|S|}{2}$.

Through out this section, the vertex set of $V(\overline{K_n \times K_2})$ is taken to be $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$ such that $\langle\{a_1, a_2, \cdots, a_n\}\rangle$, $\langle\{b_1, b_2, \cdots, b_n\}\rangle$ are null graphs and $(a_i, b_j) \in E(\overline{K_n \times K_2})$ if and only if $i \neq j$.

Lemma 2.1 *Let $S_1, S_2, \cdots, S_n$ be n efficient dominating sets of $B\mathbb{C}(\Gamma, X)$ which are mutually pairwise disjoint. Then the induced subgraph $\tilde{G} =< S_1 \cup S_2 \cup \cdots \cup S_n >$ is a $m-$fold covering graph of the graph $G = \overline{K_n \times K_2}$, where $m = \frac{|S_i|}{2}$ for each $i = 1, 2, \cdots, n$.*

Proof Note that in a graph all the efficient dominating sets have the same cardinality. Since S_1 is efficient, $S_1 = A_1 \cup B_1$ where $A_1 \subseteq (\Gamma \times 0) \cap S_1$ and $B_1 \subseteq (\Gamma \times 1) \cap S_1$ with $|A_1| = |B_1| = \frac{|S_1|}{2}$. Define $A_i = N(B_1) \cap S_i$ and $B_i = N(A_1) \cap S_i$ for $2 \leq i \leq n$. Note that $A_i \subset \Gamma \times 0$ and $B_i \subset \Gamma \times 1$ for $1 \leq i \leq n$ and $\tilde{G} =< A_1 \cup B_1 \cup A_2 \cup B_2 \cup \cdots \cup A_n \cup B_n >$.

Let $V(G) = \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$. Define $f : \tilde{G} \to G$ by $f(s) = a_i \ if \ s \in A_i$ and $f(s) = b_i \ if \ s \in B_i$ for $1 \leq i \leq n$. Let $v \in V(G)$. Suppose $v = a_i$. Then $N(v) = \{b_1, b_2, \ldots, b_{i-1}, b_{i+1}, b_{i+2}, \cdots, b_n\}$ and $f^{-1}(v) = A_i$. Let $\tilde{v} \in f^{-1}(v)$. Since S_i's are efficient, $N(\tilde{v}) = \{\beta_1, \beta_2, \cdots, \beta_{i-1}, \beta_{i+1}, \cdots, \beta_n\}$ where $\beta_j \in B_j$ for $1 \leq j \leq i-1$ and $i+1 \leq j \leq n$. By the definition of f, we have $f(\beta_j) = b_j$. Thus $f : N(\tilde{v}) \to N(v)$ is a bijection when $v = a_i$. Similarly one can prove that $f : N(\tilde{v}) \to N(v)$ is a bijection when $v = b_i$. Since $\frac{|S_i|}{2} = |A_i| = |B_i| = m$ for all $1 \leq i \leq n$, f is an $m-$fold covering of the graph $\overline{K_n \times K_2}$. $\square$

Theorem 2.2 *Let $G = B\mathbb{C}(\Gamma, X)$ and n be a positive integer. Then G is a covering graph of $\overline{K_n \times K_2}$ if and only if G has a vertex partition of n efficient dominating sets.*

Proof Suppose G is a covering of $\overline{K_n \times K_2}$. Since $\{a_i, b_i\}$ is an efficient dominating set in $\overline{K_n \times K_2}$, by Theorem 1.2, we have $f^{-1}(\{a_i, b_i\})$ is an efficient dominating set in G for $1 \leq i \leq n$. Since f is a function, $f^{-1}(\{a_i, b_i\}) \cap f^{-1}(\{a_j, b_j\}) = \emptyset$ for $i \neq j$. Hence $\{f^{-1}(\{a_i, b_i\}) : 1 \leq i \leq n\}$ is a vertex partition of efficient dominating sets in G. The other part follows from Lemma 2.1. $\square$

Lemma 2.3 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a symmetric generating set for a group Γ and let S be an efficient dominating set for the Bi-Cayley graph $G = B\mathbb{C}(\Gamma, X)$. Then we have the following:*

(a) For each $1 \leq i \leq n, S + (x_i, 0)$ is an efficient dominating set.

(b) $\{S, S + (x_1, 0), S + (x_2, 0), \cdots, S + (x_n, 0)\}$ is a vertex partition in $B\mathbb{C}(\Gamma, X)$.

Proof (a) Let $(v, 0) \in V(G)$. If $(x_i^{-1} * v, 0) \in S$, then $(v, 0) \in S + (x_i, 0)$. Suppose $(x_i^{-1} * v, 0) \notin S$. Since S is efficient, there exists unique $(s, 1) \in S$ such that $s = (x_i^{-1} * v) * x$ for some $x \in X$. That is $x_i * s = v * x$. Hence the vertex $(v, 0)$ is dominated by $(x_i * s, 1) \in S + (x_i, 0)$. Thus in all the cases we have $(v, 0) \in N[S + (x_i, 0)]$. Similarly when $(v, 1) \in V(G)$, one can prove that $(v, 1) \in N[S + (x_i, 0)]$. Thus $S + (x_i, 0)$ is a dominating set for $1 \leq i \leq n$. Since S is efficient and $|S + (x_i, 0)| = |S|$, by Theorem 1.1, we have $S + (x_i, 0)$ is an efficient dominating set for $1 \leq i \leq n$.

(b) Since S is a dominating set, for every $(u, 0) \in V(G)$, we have $(u, 0) \in S$ or $(u, 0)$ is adjacent to some vertex $(s, 1) \in S$ and so $u = s * x_i$ for some $x_i \in X$. Similar thing is holds for $(u, 1) \in V(G)$. This means that $V(G) = S \cup (S + (x_1, 0)) \cup (S + (x_2, 0)) \cup \cdots \cup (S + (x_n, 0))$. Since G is $|X|-$regular and S is an efficient dominating set, $|S| = \frac{2|\Gamma|}{|X|+1}$. That is $2|\Gamma| = (|X| + 1)|S|$. Since $|S| = |S + (x_1, 0)| = |S + (x_2, 0)| = \cdots = |S + (x_n, 0)|$, one can conclude

that $\{S, S+(x_1,0), S+(x_2,0), \cdots, S+(x_n,0)\}$ is a vertex partition of G. $\square$

From Lemmas 2.1, 2.3 one can have the following:

Corollary 2.4 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a symmetric generating set for a group Γ and let S be an efficient dominating set in $B\mathbb{C}(\Gamma, X)$. If $(x_i, 0) + S = S + (x_i, 0)$ for each $1 \leq i \leq n$, then there exist a covering $f : B\mathbb{C}(\Gamma, X) \to \overline{K_{n+1} \times K_2}$ such that $S, S+(x_1,0), S+(x_2,0), \cdots, S+(x_n,0)$ are the fibers of $\{a_i, b_i\}$ under the map f.*

Now we define the following: For $S \subset V(B\mathbb{C}(\Gamma, X))$, define $S^0 = S \cup \{(e,0)\}$.

Theorem 2.5 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a symmetric generating set for a group Γ and let M be a normal subset of Γ and $S = (M \times 0) \cup (M \times 1)$. Then the following are equivalent.*

(a) S is an efficient dominating set in $B\mathbb{C}(\Gamma, X)$.

(b) There exists a covering $f : B\mathbb{C}(\Gamma, X) \to \overline{K_{n+1} \times K_2}$ such that $f^{-1}(\{a_i, b_i\}) = S$ for some $1 \leq i \leq n$.

(c) $|S| = \frac{2|\Gamma|}{|X|+1}$ and $S \cap [S + (((X \times 0)^0 + (X \times 0)^0) - \{(e,0)\})] = \emptyset$.

Proof $(a) \Rightarrow (b)$: Since M is a normal subset, we have $(x_i, 0) + S = S + (x_i, 0)$ for $1 \leq i \leq n$ and so the proof follows from Corollary 2.4.

$(b) \Rightarrow (a)$: Since $\{a_i, b_i\}$ is an efficient dominating set in $\overline{K_n \times K_2}$, the proof follows from Theorem 1.2.

$(a) \Rightarrow (c)$: Since S is an efficient dominating set and G is $|X|-$regular, the fact $|S| = \frac{2|\Gamma|}{|X|+1}$ follows from Theorem 1.1. Suppose $S \cap [S + (((X \times 0)^0 + (X \times 0)^0) - \{(e,0)\})] \neq \emptyset$. Then there exist $(s,0)$(or $(s,1)$) $\in S$ such that $(s,0) = (s_1,0) + (x,0) + (x_1,0)$ with $x, x_1 \in X, x \neq x_1^{-1}$ and $(s_1, 0)$ (or $(s_1,1)$) $\in S$. Since $x \neq x_1^{-1}$, we have $s \neq s_1$. Thus $s * x^{-1} = s_1 * x_1$ and so $(s_1 * x_1, 1)$ is adjacent to two vertices $(s,0), (s_1,0) \in S$, a contradiction to S is efficient.

$(c) \Rightarrow (a)$: Let $x_i, x_j \in X$ with $x_i \neq x_j$. Suppose $(S + (x_i,0)) \cap (S + (x_j,0)) \neq \emptyset$. Let $a \in (S + (x_i,0)) \cap (S + (x_j,0))$. Then $a = (s_1,0) + (x_i,0) = (s_2,0) + (x_j,0)$ or $(s_1,1) + (x_i,0) = (s_2,1) + (x_j,0)$. Hence $s_1 * x_i = s_2 * x_j$ and so $s_1 = s_2 * x_j * x_i^{-1}$. Since $x_i \neq x_j$, we have $x_i^{-1} * x_j \neq e$. Thus $(s_1, 0) \in S \cap [S + (((X \times 0)^0 + (X \times 0)^0) - \{(e,0)\})]$, a contradiction. Suppose $S \cap (S + (x,0)) \neq \emptyset$ for some $x \in X$. Then $(s,0) = (s_1,0) + (x,0)$ or $(s,1) = (s_1,1) + (x,0)$. Thus $(s,0) = (s_1,0) + (x,0) + (e,0)$ or $(s,1) = (s_1,1) + (x,0) + (e,0)$. Since $x \neq e$, $(s,0) \in S \cap [S + (((X \times 0)^0 + (X \times 0)^0) - \{(e,0)\})]$, a contradiction. Thus $S \cup \{S + (x_i, 0) : 1 \leq i \leq n\}$ is a collection of pairwise disjoint sets. Now $N[S] = \bigcup\limits_{s \in S} N[s] = \bigcup\limits_{s \in S} [s + (X \times 0)^0] = \bigcup\limits_{(x,0) \in (X \times 0)^0} [S + (x,0)]$. Since $|S + (x,0)| = |S|$, $|N[S]| = |S||(X \times 0)^0| = |S|(|X|+1) = |S|(\frac{2|\Gamma|}{|S|}) = 2|\Gamma|$. Thus S is a dominating set. Since $|S| = \frac{2|\Gamma|}{|X|+1}$, by Theorem 1.1, S is an efficient dominating set. $\square$

§3. Efficient Open Domination and Bi-Cayley Graphs

In this section, we find the necessary and sufficient condition for the existence of an efficient open dominating set in $B\mathbb{C}(\Gamma, X)$. Note that if S is an efficient open dominating set of a graph

G, then $\{N(v) : v \in S\}$ is a partition of $V(G)$ and if G has an efficient open dominating set, then all efficient open dominating sets in G have the same cardinality namely $\gamma_t(G)$.

Through out this section, the vertex set of $K_{n,n}$ is taken as $\{c_1, c_2, \cdots, c_n, d_1, d_2, \cdots, d_n\}$ where no two c_i's are adjacent and no two d_i's are adjacent.

Remark 3.1 If S is an efficient open dominating set in $G = B\mathbb{C}(\Gamma, X)$, then $|S|$ is even and we can write $S = C \cup D$ where $|C| = |D| = \frac{|S|}{2}$ and every edge of $\langle C \cup D\rangle$ has one end in C and another end in D. Note that if G is a $k-$regular graph, then $\gamma_t(G) \geq \frac{|V(G)|}{k}$ and equality holds if and only if G has an efficient open dominating set.

Lemma 3.1 *Let $S_1, S_2, \cdots, S_n$ be n mutually pairwise disjoint efficient open dominating sets of $B\mathbb{C}(\Gamma, X)$. Then the induced subgraph $\tilde{G} = \langle S_1 \cup S_2 \cup \cdots \cup S_n\rangle$ is a $m-$fold covering graph of $G = K_{n,n}$, where $m = \frac{|S_i|}{2}$ for each $i = 1, 2, \cdots, n$.*

Proof Since S_i is efficient open for each $1 \leq i \leq n$, we have $S_i = C_i \cup D_i$ where $C_i \subseteq (\Gamma \times 0) \cap S_i$ and $D_i \subseteq (\Gamma \times 1) \cap S_i$ with $|C_i| = |D_i| = \frac{|S_i|}{2}$ and every edge in the induced subgraph $\langle C_i \cup D_i\rangle$ has one end in C_i and other in D_i. Note that $\tilde{G} = \langle C_1 \cup D_1 \cup C_2 \cup D_2 \cup \cdots \cup C_n \cup D_n\rangle$. Let $V(G) = \{c_1, c_2, \cdots, c_n, d_1, d_2, \cdots, d_n\}$.

Define $f : \tilde{G} \to G$ by $f(s) = c_i$ if $s \in C_i$ and $f(s) = d_i$ if $s \in D_i$ for $1 \leq i \leq n$. Let $v \in V(G)$. Suppose $v = c_i$. Then $N(v) = \{d_1, d_2, \ldots, d_n\}$ and $f^{-1}(v) = C_i$. Let $\tilde{v} \in f^{-1}(v)$. Since S_i's are efficient open, $N(\tilde{v}) = \{\beta_1, \beta_2, \cdots, \beta_n\}$ where $\beta_j \in D_j$ for $1 \leq j \leq n$. By the definition of f, we have $f(\beta_j) = d_j$. Thus $f : N(\tilde{v}) \to N(v)$ is a bijection when $v = c_i$. Similarly one can prove that $f : N(\tilde{v}) \to N(v)$ is a bijection when $v = d_i$. Since $\frac{|S_i|}{2} = |C_i| = |D_i| = m$ for all $1 \leq i \leq n$, f is an $m-$fold covering of the graph $K_{n,n}$. $\square$

Remark 3.3 Let $f : \tilde{G} \to G$ be a covering and S be an efficient open dominating set of G. By the definition of an efficient open domination, S is perfect and so by Theorem 1.2, $f^{-1}(S)$ is perfect. That is $|N(\tilde{v}) \cap f^{-1}(S)| = 1$ for all $\tilde{v} \in \tilde{G} - f^{-1}(S)$. Let $\tilde{v} \in f^{-1}(S)$. Then $f(\tilde{v}) = v \in S$. Since S is an efficient open dominating set, there exist unique $w \in S$ such that v and w are adjacent. Since $f|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection, $\tilde{w} = f^{-1}(w)$ is the only vertex adjacent to $\tilde{v}$ in $f^{-1}(S)$. That is $|N(\tilde{v}) \cap f^{-1}(S)| = 1$ for all $\tilde{v} \in f^{-1}(S)$. Hence inverse image of an efficient open dominating set under a covering function is an efficient open dominating set.

Theorem 3.4 *Let $G = B\mathbb{C}(\Gamma, X)$ and n be a positive integer. Then G is a covering of $K_{n,n}$ if and only if G has a vertex partition of efficient open dominating sets.*

Proof Suppose G is a covering graph of $K_{n,n}$. Since the pair $\{c_i, d_i\}$ is an efficient open dominating set in $K_{n,n}$, by Remark 3.3, $f^{-1}(\{c_i, d_i\})$ is an efficient open dominating set in G for $1 \leq i \leq n$. Since f is a function, $\{f^{-1}(\{c_i, d_i\}) : 1 \leq i \leq n\}$ is a partition of efficient open dominating sets in G. The other part follows from Lemma 3.2. $\square$

Lemma 3.5 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a symmetric generating set for a group Γ and let S be an efficient open dominating set for the Bi-Cayley graph $G = B\mathbb{C}(\Gamma, X)$. Then we have the following:*

(a) For each $1 \le i \le n, S + (x_i, 0)$ is an efficient open dominating set.

(b) $\{S + (x_1, 0), S + (x_2, 0), \cdots, S + (x_n, 0)\}$ is a vertex partition of $B\mathbb{C}(\Gamma, X)$.

Proof (a) Let $(v, 0) \in V(G)$. Consider the vertex $(x_i^{-1} * v, 0) \in V(G)$. Since S is an open dominating set, there exists $(s, 1) \in S$ such that $s = (x_i^{-1} * v) * x$ for some $x \in X$. That is $x_i * s = v * x$. Hence the vertex $(v, 0)$ is dominated by $(x_i * s, 1) \in S + (x_i, 0)$ and so $(v, 0) \in N(S+(x_i, 0))$. Similarly when $(v, 1) \in V(G)$, one can prove that $(v, 1) \in N(S+(x_i, 0))$. Thus $S+(x_i, 0)$ is an open dominating set for $1 \le i \le n$. Since S is an efficient open dominating set and $|S| = |S + (x_i, 0)|$, by Remark 3.1, $S + (x_i, 0)$ is an efficient open dominating set for $1 \le i \le n$.

(b) Since S is an open dominating set, for every $(u, 0) \in V(G)$ there exists $(s, 1) \in S$ such that $u = s * x_i$ for some $x_i \in X$. Similar thing is holds for $(u, 1) \in V(G)$. This means that $V(G) = (S + (x_1, 0)) \cup (S + (x_2, 0)) \cup \cdots \cup (S + (x_n, 0))$. Since G is $|X|-$regular and S is an efficient open dominating set, $|S| = \frac{2|\Gamma|}{|X|}$. That is $2|\Gamma| = |X|\ |S|$. Since $|S| = |S + (x_1, 0)| = |S+(x_2, 0)| = \cdots = |S+(x_n, 0)|$, one can conclude that $\{S, S+(x_1, 0), S+(x_2, 0), \cdots, S+(x_n, 0)\}$ is a vertex partition of G. $\square$

From the proof of Lemma 3.2 and by Lemma 3.5, the following corollary follows:

Corollary 3.6 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a symmetric generating set for a group Γ and let S be an efficient dominating set in $B\mathbb{C}(\Gamma, X)$. If $(x_i, 0) + S = S + (x_i, 0)$ for each $1 \le i \le n$, then there exists a covering $f : B\mathbb{C}(\Gamma, X) \to K_{n,n}$ such that $S + (x_1, 0), S + (x_2, 0), \cdots, S + (x_n, 0)$ are the fibers of $\{c_i, d_i\}$ under the map f.*

Theorem 3.7 *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a symmetric generating set for a group Γ, M be a normal subset of Γ and $S = (M \times 0) \cup (M \times 1)$. Then the following are equivalent.*

(a) S is an efficient open dominating set in $B\mathbb{C}(\Gamma, X)$.

(b) There exists a covering $f : B\mathbb{C}(\Gamma, X) \to K_{n,n}$ such that $f^{-1}(\{c_i, d_i\}) = S$ for some $1 \le i \le n$.

(c) $|S| = \frac{2|\Gamma|}{|X|}$ and $S \cap [S + (((X \times 0) + (X \times 0)) - \{(e, 0)\})] = \emptyset$.

Proof $(a) \Rightarrow (b)$: Proof follows from Corollary 3.6.

$(b) \Rightarrow (a)$: Since $\{c_i, d_i\}$ is an efficient open dominating set in $K_{n,n}$, the proof follows from Remark 3.3.

$(a) \Rightarrow (c)$: Since S is an efficient open and G is $|X|-$regular, the fact $|S| = \frac{2|\Gamma|}{|X|}$ follows from Remark 3.1. Suppose $S \cap [S + (((X \times 0) + (X \times 0)) - \{(e, 0)\})] \ne \emptyset$. Then there exist $(s, 0)$(or $(s, 1)$) $\in S$ such that $(s, 0) = (s_1, 0) + (x, 0) + (x_1, 0)$ with $x, x_1 \in X, x \ne x_1^{-1}$ and $(s_1, 0)$ (or $(s_1, 1)$) $\in S$. Since $x \ne x_1^{-1}$, we have $s \ne s_1$. Since $s = s_1 * x * x_1$, we have $s * x^{-1} = s_1 * x_1$ and so $(s_1 + x_1, 1)$ is adjacent with two vertices $(s, 0), (s_1, 0) \in S$, a contradiction to S is efficient open.

$(c) \Rightarrow (a)$: Let $x_i, x_j \in X$ with $x_i \ne x_j$. Suppose $(S + (x_i, 0)) \cap (S + (x_j, 0)) \ne \emptyset$. Let $a \in (S+(x_i, 0)) \cap (S+(x_j, 0))$. Then $a = (s_1, 0)+(x_i, 0) = (s_2, 0)+(x_j, 0)$ or $a = (s_1, 1)+(x_i, 0) = (s_2, 1) + (x_j, 0)$. Since $x_i * x_j^{-1} \ne e$, $(s_1, 0) \in S \cap [S + (((X \times 0) + (X \times 0)) - \{(e, 0)\})]$, a

contradiction. Thus $\{S + (x_i, 0) : 1 \leq i \leq n\}$ is a collection of pairwise disjoint sets. Now $N(S) = \bigcup_{s \in S} N(s) = \bigcup_{s \in S} [s + (X \times 0)] = \bigcup_{(x,0) \in (X \times 0)} [S + (x, 0)]$. Since $|S + (x_i, 0)| = |S|$, $|N(S)| = |S||(X \times 0)| = |S||X| = |S|(\frac{2|\Gamma|}{|S|}) = 2|\Gamma|$. Thus S is an open dominating set. Since $|S| = \frac{2|\Gamma|}{|X|}$, one can conclude that S is an efficient open dominating set. $\square$

Acknowledgment

The work reported here is supported by the Project $SR/S4/MS : 328/06$, awarded to authors by the Department of Science and Technology, Government of India, New Delhi.

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Math.Combin.Book.Ser. Combin. Vol.4 (2009), 63-74

Independent Complementary Distance Pattern Uniform Graphs

Germina K.A. and Beena Koshy

PG and Research Department of Mathematics of Kannur University

Mary Matha Arts & Science College, Vemom P.O., Mananthavady - 670645, India

E-mail: srgerminaka@gmail.com, beenakoshy1@yahoo.co.in

Abstract: A graph $G = (V, E)$ is called to be *Smarandachely uniform k-graph* for an integer $k \geq 1$ if there exists $M_1, M_2, \cdots, M_k \subset V(G)$ such that $f_{M_i}(u) = \{d(u, v) : v \in M_i\}$ for $\forall u \in V(G) - M_i$ is independent of the choice of $u \in V(G) - M_i$ and integer i, $1 \leq i \leq k$. Each such set M_i, $1 \leq i \leq k$ is called a CDPU set [6, 7]. Particularly, for $k = 1$, a Smarandachely uniform 1-graph is abbreviated to a *complementary distance pattern uniform* graph, i.e., CDPU graphs. This paper studies independent CDPU graphs.

Key Words: Smarandachely uniform k-graph, complementary distance pattern uniform, independent CDPU.

AMS(2000): 05C22.

§1. Introduction

For all terminology and notation in graph theory, not defined specifically in this paper, we refer the reader to Harary [4]. Unless mentioned otherwise, all the graphs considered in this paper are simple, self-loop-free and finite.

Let $G = (V, E)$ represent the structure of a chemical molecule. Often, a topological index (TI), derived as an invariant of G, is used to represent a chemical property of the molecule. There are a number of TIs based on distance concepts in graphs [5] and some of them could be designed using distance patterns of vertices in a graph. There are strong indications in the literature cited above that the notion of CDPU sets in G could be used to design a class of TIs that represent certain stereochemical properties of the molecule.

Definition 1.1([6]) *Let $G = (V, E)$ be a (p, q) graph and M be any non-empty subset of $V(G)$. Each vertex u in G is associated with the set $f_M(u) = \{d(u, v) : v \in M\}$, where $d(u, v)$ denotes the usual distance between u and v in G, called the M-distance pattern of u.*

[1]Received Oct.9, 2009. Accepted Nov. 24, 2009.

A graph $G = (V, E)$ is called to be Smarandachely uniform k-graph for an integer $k \geq 1$ if there exists $M_1, M_2, \cdots, M_k \subset V(G)$ such that $f_{M_i}(u) = \{d(u, v) : v \in M_i\}$ for $\forall u \in V(G) - M_i$ is independent of the choice of $u \in V(G) - M_i$ and integer i, $1 \leq i \leq k$. Each such set M_i, $1 \leq i \leq k$ is called a CDPU set. Particularly, for $k = 1$, a Smarandachely uniform 1-graph is abbreviated to a complementary distance pattern uniform graph, i.e., CDPU graphs. The least cardinality of the CDPU set is called the CDPU number denoted by $\sigma(G)$.

The following are some of the results used in this paper.

Theorem 1.2([7]) *Every connected graph has a CDPU set.*

Definition 1.3([7]) *The least cardinality of CDPU set in G is called the* CDPU number *of G, denoted $\sigma(G)$.*

Remark 1.4([7]) Let G be a connected graph of order p and let $(e_1, e_2, \ldots, e_k)$ be the non decreasing sequence of eccentricities of its vertices. Let M consists of the vertices with eccentricities $e_1, e_2, \ldots, e_{k-1}$ and let $|V - M| = p - m$ where $|M| = m$. Then $\sigma(G) \leqslant m$, since all the vertices in $V - M$ have $f_M(v) = \{1, 2, \ldots, e_{k-1}\}$.

Theorem 1.5([7]) *A graph G has $\sigma(G) = 1$ if and only if G has at least one vertex of full degree.*

Corollary 1.6([7]) *For any positive integer n, $\sigma(G + K_m) = 1$.*

Theorem 1.7([7]) *For any integer n, $\sigma(P_n) = n - 2$.*

Theorem 1.8([7]) *For all integers $a_1 \geq a_2 \geq \cdots \geq a_n \geq 2$, $\sigma(K_{a_1, a_2, \ldots, a_n}) = n$.*

Theorem 1.9([7]) *$\sigma(C_n) = n - 2$, if n is odd and*
$\sigma(C_n) = n/2$, if $n \geq 8$ is even. Also $\sigma(C_4) = \sigma(C_6) = 2$.

Theorem 1.10([7]) *If $\sigma(G_1) = k_1$ and $\sigma(G_2) = k_2$, then $\sigma(G_1 + G_2) = min(k_1, k_2)$.*

Theorem 1.11([7]) *Let T be a CDPU tree. Then $\sigma(T) = 1$ if and only if T is isomorphic to P_2, P_3 or $K_{1,n}$.*

Theorem 1.12([7]) *The central subgraph of a maximal outerplanar graph has CDPU number* 1 *or* 3.

Remark 1.13([7]) For a graph G which is not self centered, $\max f_M(v) = \mathrm{diam}(G) - 1$.

Theorem 1.14([7]) *The shadow graph of a complete graph K_n has exactly two $\sigma(K_n)$ disjoint CDPU sets.*

The following were the problems identified by B. D. Acharya [6, 7].

Problem 1.15 *Characterize graphs G in which every minimal CDPU-set is independent.*

Problem 1.16 *What is the maximum cardinality of a minimal CDPU set in G.*

Problem 1.17 *Determine whether every graph has an independent CDPU-set.*

Problem 1.18 *Characterize minimal CDPU-set.*

Fig.1 following depicts an independent CDPU graph.

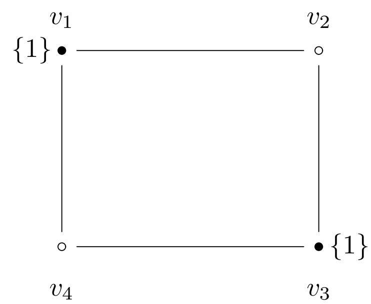

Fig.1: An independent CDPU graph with $M = \{v_2, v_4\}$

§2. Main Results

Definition 2.1 *A graph G is called an Independent CDPU graph if there exists an independent CDPU set for G.*

Following two observations are immediate.

Observations 2.2 Complete graphs are independent CDPU.

Observations 2.3 Star graph $K_{1,n}$ is an Independent CDPU graph.

Proposition 2.4 *C_n with n even is an Independent CDPU graph.*

Proof Let C_n be a cycle on n vertices and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$, where n is even. Choose M as the set of alternate vertices on C_n, say, $\{v_2, v_4, \ldots, v_n\}$. Then,
$f_M(v_i) = \{1, 3, 5, \ldots, m-1\}$ for $i = 1, 3, \ldots, n-1$, if $C_n = 2m$ and m is even and
$f_M(v_i) = \{1, 3, 5, \ldots, m\}$, for $i = 1, 3, \ldots, n-1$ if $C_n = 2m$ and m odd. Therefore, $f_M(v_i)$ is identical depending on whether m is odd or even. Hence, the alternate vertices $\{v_2, v_4, \ldots, v_n\}$ forms a CDPU set M. Also all the vertices in M are non-adjacent. Hence C_n, n even is an independent CDPU graph. □

Theorem 2.5 *A cycle C_n is an independent CDPU graph if and only if n is even.*

Proof Let C_n be a cycle on n vertices. Suppose n is even. Then from Proposition 2.4, C_n is an independent CDPU graph.

Conversely, suppose that C_n is an independent CDPU graph. That is, there exist vertices in M such that every pair of vertices are non adjacent. We have to prove that n is even. Suppose n is odd. Then from Theorem 1.9, $\sigma(C_n) = n - 2$, which implies that $|M| \geq n - 2$.

But from n vertices, we cannot have $n-2$ (or more) vertices which are non-adjacent. □

Theorem 2.6 *A graph G which contains a full degree vertex is an independent CDPU.*

Proof Let G be a graph which contains a full degree vertex v. Then, from Theorem 1.5, G is CDPU with CDPU set $M = \{v\}$. Also M is independent. Therefore, G is an independent CDPU. □

Remark 2.7 If the CDPU number of a graph G is 1, then clearly G is independent CDPU.

Theorem 2.8 *A complete n-partite graph G is an independent CDPU graph for any n.*

Proof Let $G = K_{a_1,a_2,\ldots,a_n}$ be a complete n-partite graph. Then, $V(G)$ can be partitioned into n subsets $V_1, V_2, \ldots, V_n$ where $|V_1| = a_1, |V_2| = a_2, \ldots, |V_n| = a_n$. Take all the vertices from the partite set, say, V_i of $K_{a_1,a_2,\ldots,a_n}$ to constitute the set M. Since each element of a partite set is non-adjacent to the other vertices in it and is adjacent to all other partite sets, we get, $f_M(u) = \{1\}, \forall u \in V(K_{a_1,a_2,\ldots,a_n}) - M$. Hence, the complete n-partite graph G is an independent CDPU graph for any n. □

Corollary 2.9 *Complete n-partite graphs have n distinct independent CDPU sets.*

Proof Let $G = K_{a_1,a_2,\ldots,a_n}$ be a complete n-partite graph. Then, $V(G)$ can be partitioned into n subsets $V_1, V_2, \ldots, V_n$ where $|V_1| = a_1, |V_2| = a_2, \ldots, |V_n| = a_n$. Take M_1 as the vertices corresponding to the partite set V_1, M_2 as the vertices corresponding to the partite set V_2, $\ldots$, M_i corresponds to the vertices of the partite set V_i, $\ldots$, M_n corresponds to the vertices of the partite set V_n. Then from Theorem 2.8, each $M_i, 1 \leq i \leq n$ form a CDPU set. Hence there are n distinct CDPU sets. □

Theorem 2.10 *A path P_n is an independent CDPU graph if and only if $n = 2, 3, 4, 5$.*

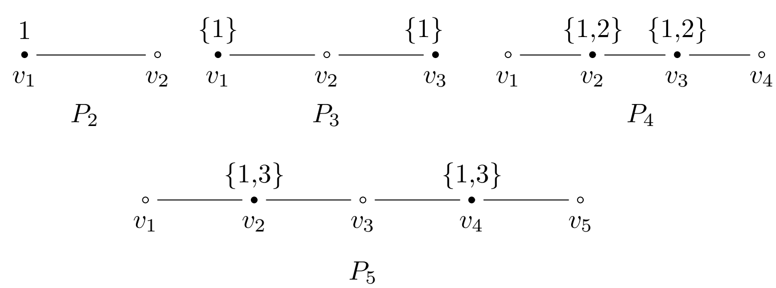

Fig.2: An independent CDPU paths

Proof Let P_n be a path on n vertices and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. When $n = 2$ and 3, P_2 and P_3 contains a vertex of full degree and hence from Theorem 2.6, P_2 and P_3 are independent

CDPU. When $n = 4$, take $M = \{v_1, v_4\}$. Then $f_M(v_2) = f_M(v_3) = \{1, 2\}$, whence M is independent CDPU. When $n = 5$, let $V(G) = \{v_1, v_2, \ldots, v_5\}$ and choose $M = \{v_1, v_3, v_5\}$. Then, $f_M(v_2) = f_M(v_4) = \{1, 3\}$. Hence, P_5 is an independent CDPU graph.

Conversely, suppose that P_n is an independent CDPU graph. That is, there exists a CDPU set M such that no two of the vertices are adjacent. From n vertices, we can have at most $\frac{n}{2}$ or $\frac{n+1}{2}$ vertices which are non adjacent. From Theorem 1.7, $\sigma(P_n) = n - 2, n \geq 3$. When $n \geq 6$, we cannot choose a CDPU set M such that $n - 2$ vertices are non-adjacent. Hence P_n is independent CDPU only for $n = 2, 3, 4$ and 5. $\square$

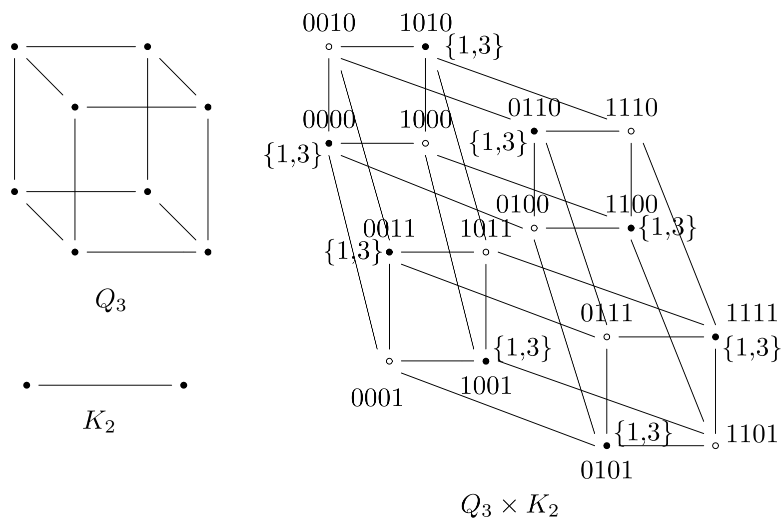

Fig.3 : Q_4

Theorem 2.11 *n-cube Q_n is an independent CDPU graph with $|M| = 2^{n-1}$.*

Proof We have $Q_n = K_2 \times Q_{n-1}$ and has 2^n vertices which may be labeled $a_1 a_2 \ldots a_n$, where each a_i is either 0 or 1. Also two points in Q_n are adjacent if their binary representations differ at exactly one place. Take M as the set of all vertices whose binary representation differ at two places. Clearly the vertices in M are non adjacent and also maximal. We have to check whether M is CDPU. For let $M = \{v_1, v_3, \ldots, v_{2^n - 1}\}$. Consider a vertex v_i which does not belong to M. Clearly v_i is adjacent to a vertex v_j in M. Hence $1 \in f_M(v_i)$. Then, since v_j is in M, v_j is adjacent to a vertex v_k not in M. Hence 2 does not belong to $f_M(v_i)$. Since v_k is not an element of M and v_k is adjacent to a vertex v_l in M, $3 \in f_M(v_i)$. Proceeding in the same manner, we get $f_M(v_i) = \{1, 3, \ldots, n - 1\}$. Hence Q_n is independent CDPU with $|M| = \frac{2^n}{2}$. $\square$

Theorem 2.12 *Ladder $P_n \times K_2$ is an independent CDPU graph if and only if $n \leq 4$.*

Proof First we have to prove that $P_n \times K_2$ is an independent CDPU graph for $n \leq 4$.

When $n = 2$, take $M = \{v_2, v_4\}$, so that $f_M(v_i) = \{1\}$ for $i = 1, 3$.

When $n=3$, take $M=\{v_1,v_4\}$, so that $f_M(v_i)=\{1,2\}$, for $i=2,4,6$.

When $n=4$, take $M=\{v_1,v_3,v_5,v_7\}$, so that $f_M(v_i)=\{1,3\}$ for $i=2,4,6,8$. Therefore, $P_n\times K_2$ is an independent CDPU graph for $n\le 4$.

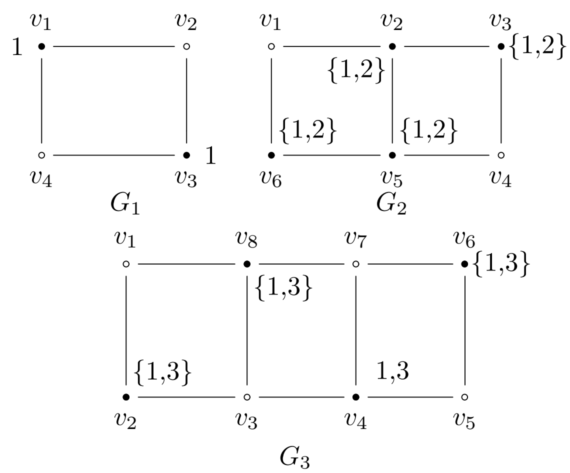

Fig.4: $P_n\times K_2$ for $n\le 4$

Conversely, suppose that $P_n\times K_2$ is an independent CDPU graph. We have to prove that $n\le 4$. If possible, suppose $n=k\ge 5$. In $P_n\times K_2$, since the number of vertices is even, and the vertices in $P_n\times K_2$ forms a Hamiltonian cycle, then the only possibility of M to be an independent CDPU set is to choose M as the set of all alternate vertices of the Hamiltonian cycle. Clearly, in this case M is a maximal independent set. Denote $M_1=\{v_1,v_3,\ldots,v_{2n-1}\}$ and $M_2=\{v_2,v_4,\ldots,v_{2n}\}$. Consider $M_1=\{v_2,v_4,\ldots,v_i,\ldots,v_{2n}\}$.

Case 1 n is odd.

In this case, $f_{M_1}(v_1)=\{1,3,\ldots,n\}$. Since n is odd we have two central vertices, say, v_i and v_j in $P_n\times K_2$. Since v_i and v_j are of the same eccentricity and M_1 is a maximal independent set, v_j does not belong to M_1. Then, $f_{M_1}(v_j)=\{1,3,\ldots,\frac{n+1}{2}\}$.

Thus, $f_{M_1}(v_1)\ne f_{M_1}(v_j)$. Hence M_1 is not a CDPU.

Case 2 n is even.

In this case, $f_{M_1}(v_1)=\{1,3,\ldots,n-1\}$. Since n is even, there are four central vertices v_i,v_j,v_k,v_l in $P_n\times K_2$. Clearly the graph induced by $T=\{v_i,v_j,v_k,v_l\}$ is a cycle on four vertices. Since M_1 is maximal and consists of the alternate vertices of $P_n\times K_n$, v_j,v_l should necessarily be outside M_1. Thus, $f_{M_1}(v_j)=\{1,3,\ldots,\frac{n}{2}\}$.

Thus, $f_{M_1}(v_1)\ne f_{M_1}(v_j)$. Hence M_1 is not a CDPU.

Therefore $P_n\times K_2$ is not independent CDPU for $n\ge 5$. Hence the theorem. □

Theorem 2.13 *If G_1 and G_2 are independent CDPU graphs, then G_1+G_2 is also an independent CDPU graph.*

Proof Since G_1 and G_2 are independent CDPU graphs, there exist $M_1 \subset V(G_1)$ and $M_2 \subset V(G_2)$ such that no two vertices in M_1 (and in M_2) are adjacent. Now, in G_1+G_2, every vertex of G_1 is adjacent to every vertices of G_2. Then clearly, independent CDPU set M_1 of G_1 (or M_2 of G_2) is an independent CDPU set for G_1+G_2. Hence the theorem. □

Remark 2.14 If G_1 and G_2 are independent CDPU graphs, then the cartesian product $G_1 \times G_2$ need not have an independent CDPU set. But $G_i \times G_i$ is independent CDPU for $i = 1, 2$ as illustrated in Fig.5.

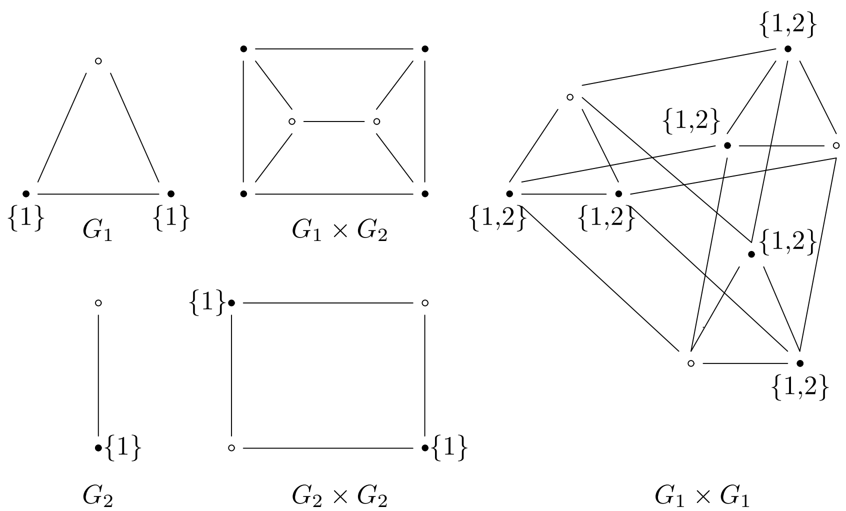

Fig.5

Definition 2.15 *An independent set that is not a proper subset of any independent set of G is called maximal independent set of G. The number of vertices in the largest independent set of G is called the independence number of G and is denoted by $\beta(G)$.*

§3. Independence CDPU Number

The least cardinality of the independent cdpu set in G is called the independent CDPU number of G, denoted by $\sigma_i(G)$. In general, for an independent CDPU graph, $\sigma_i(G) \leq \beta(G)$, where $\beta(G)$ is the independence number of G.

Theorem 3.1 *If G is an independent CDPU graph with n vertices, then $r(G) \leq \sigma_i(G) \leq \lceil \frac{n}{2} \rceil$, where $r(G)$ is the radius of G.*

Proof We have, $\beta(G) \leq \lceil \frac{n}{2} \rceil$ and hence $\sigma_i(G) \leq \lceil \frac{n}{2} \rceil$. Now we prove that $r(G) \leq \sigma_i(G)$. Suppose $r(G) = k$. Then, there are vertices with eccentricities $k, k+1, k+2, \ldots, d$, where d is the diameter of G. Let v be the central vertex of G and $e = uv$. Since the central vertex v of a graph on $n(\geq 3)$ vertices cannot be a pendant vertex, there exists a vertex w which is adjacent to v. Hence, w is of eccentricity $k+1$. Also u is of eccentricity $k+1$. By a similar

argument there exists at least two vertices each of eccentricity $k+1, k+2, \ldots, d$. Hence, the CDPU set should necessarily consists of all vertices with eccentricity $k, k+1, k+2, \ldots, d-1$. Thus, $\sigma(G) \geq 1 + \{2 + 2 + \ldots (d-1-k) times\} \geq k$. Whence, $\sigma_i(G) \geq r(G)$. Therefore, $r(G) \leq \sigma_i(G) \leq \lceil \frac{n}{2} \rceil$. $\square$

Theorem 3.2 *A graph G has $\sigma_i(G) = 1$ if and only if G has at least one vertex of full degree.*

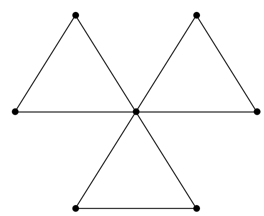

Fig.6: A graph with $\sigma_i(G) = 1$

Proof Suppose that G has one vertex v_i with full degree. Take $M = \{v_i\}$. Then $f_M(u) = \{1\}$, for every $u \in V - M$. Also M is independent. Hence $\sigma_i(G) = 1$.

Conversely, suppose that G is a graph with $\sigma_i(G) = 1$. That is, there exists an independent set M which contains only one vertex v_i which is a CDPU set of G. Also, $\sigma_i(G) = 1$ implies, v_i is adjacent to all other vertices. Hence v_i is a vertex with full degree. $\square$

Corollary 3.3 *The independent CDPU number of a complete graph is* 1.

Corollary 3.4 *If M is the maximal independent set of a graph G with $|M| = 1$, then G is an independent CDPU.*

Proof The result follows since M is a maximal independent set and $|M| = 1$, there is a vertex v of full degree. $\square$

Theorem 3.5 *Peterson Graph is an independent CDPU graph with $\sigma_i(G) = 4$.*

Proof Let G be a Peterson Graph with $V(G) = \{v_1, v_2, \ldots, v_{10}\}$. Let M be such that M contains two non adjacent vertices from the outer cycle and two non-adjacent vertices from the inner cycle. Let it be $\{v_3, v_5, v_6, v_7\}$. Clearly, M is a maximal independent set of G. Also $f_M(v_i) = \{1, 2\}$, for every $i = 1, 2, 4, 8, 9, 10$. Thus, M is a CDPU set of G. Hence, G is an independent CDPU graph with $\sigma_i(G) \leq 4$. To prove that $\sigma_i(G) = 4$, it is enough to prove that the deletion of any vertex from M does not form a CDPU set. For, let $M_1 = \{v_3, v_5, v_7\}$. Then, $f_M(v_i) = \{1, 2\}$, for $i = 1, 2, 4, 8, 9, 10$ and $f_M(v_6) = \{2\}$. Hence M_1 cannot be a CDPU set for G. Thus $\sigma_i(G) = 4$. $\square$

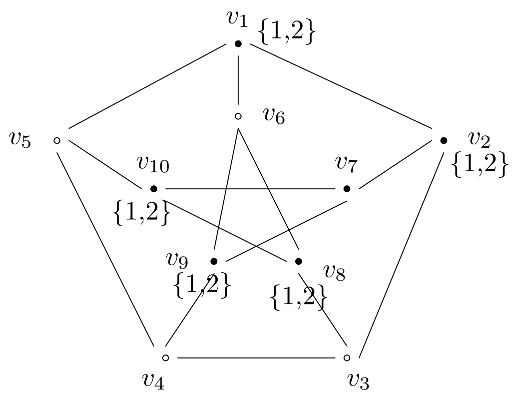

Fig.7

Theorem 3.6 *Shadow graphs of K_n are independent CDPU with $|M| = n$.*

Proof Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n and $v_1', v_2', \ldots, v_n'$ be the corresponding shadow vertices. Clearly, $M = \{v_1', v_2', \ldots, v_n'\}$ is a maximal independent set of $S(K_n)$. Also, from Theorem 1.14, M forms a CDPU set. Hence $|M| = n$. $\Box$

Definition 3.7 *A set of points which covers all the lines of a graph G is called a point cover for G. The smallest number of points in any point cover for G is called its point covering number and is denoted by $\alpha_0(G)$.*

It is natural to rise the following question by definition:

Does there exist any connection between the point covering for a graph and independent CDPU set?

Proposition 3.8 *If $\alpha_0(G) = 1$, then $\sigma_i(G) = 1$*

Proof Since $\alpha_0(G) = 1$, we have to cover every edges by a single vertex. This implies that there exists a vertex of full degree. Hence from Theorem 3.2, $\sigma_i(G) = 1$. $\Box$

Remark 3.9 The converse of Proposition 3.8 need not be true. Note that in Figure 6, $\sigma_i(G) = 1$, but $\alpha_0(G) = 6$.

Theorem 3.10 *The central subgraph $< C(G) >$ of a maximal outerplanr graph G is an independent CDPU graph with $\sigma_i(G) = 1, 2$ or 3.*

Proof Fig.8 depicts all the central subgraphs of maximal outerplanr graph [3]. Since G_1, G_2, G_3, G_4, G_5 have a full degree vertex, those graphs are independent CDPU and $\sigma_i(G_j) = 1$, for $j = 1, 2, 3, 4, 5$.

In G_6, let $M = \{v_1, v_4\}$. Then, $f_M(v_i) = \{1, 2\}$, for every $v_i \in V - M$. Since M is independent, G_6 is independent CDPU and $\sigma_i(G_6) = 2$.

In G_7, let $M = \{v_1, v_3, v_5\}$. Then, $f_M(v_i) = \{1, 2\}$ for every $v_i \in V - M$. Hence, G_7 is independent CDPU with $\sigma_i(G_7) = 3$. □

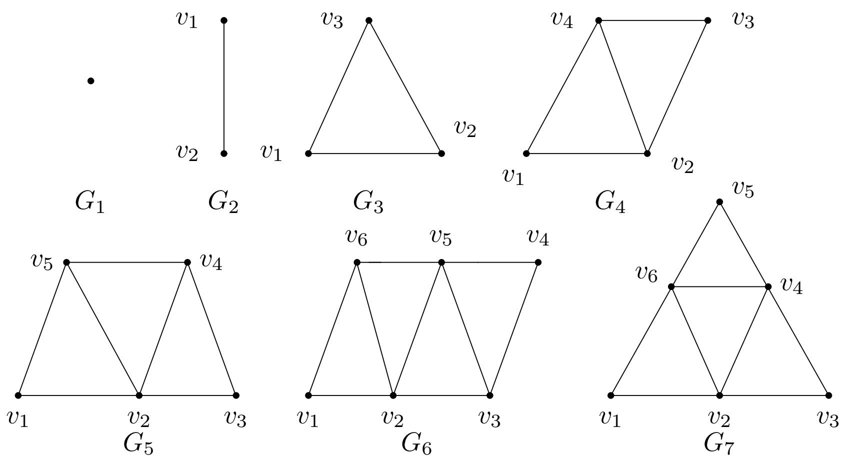

Fig.8: Central subgraphs of a maximal outerplanar graph

Theorem 3.11 *The independent CDPU number of an even cycle* $C_n, n \geq 8$ *is* $\frac{n}{2}$.

Proof From Proposition 2.4, the alternate vertices of the even cycle constitute the independent CDPU set. As already proved, removal of any vertex from M does not give a cdpu set. Hence, $\sigma_i(C_n) = \frac{n}{2}$. □

Remark 3.12 $\sigma_i(C_6) = 2$.

Theorem 3.13 *For all integers* $a_1 \geq a_2 \geq \cdots \geq a_n \geq 2, \sigma_i(K_{a_1,a_2,\ldots,a_n}) = min\{a_1, a_2, \ldots, a_n\}$.

Proof From Theorem 2.8 and Corollary 2.9, all the n partite sets form an independent CDPU set. Hence the independent CDPU number is the minimum of all $a_i's$. □

Theorem 3.14 *If* $\sigma_i(G_1) = k_1$ *and* $\sigma_i(G_2) = k_2$, *then* $\sigma_i(G_1 + G_2) = min.\{k_1, k_2\}$.

Proof From Theorem 2.13, either M_1 or M_2 is an independent cdpu set for $G_1 + G_2$. Also $\sigma_i(G_1 + G_2)$ is the minimum among M_1 and M_2. □

Theorem 3.15 *If* G_1 *and* G_2 *are independent CDPU cycles with* $n, m(\geq 4)$ *vertices respectively, then* $G_1 \times G_2$ *is independent CDPU with* $|M| = \frac{mn}{2}$.

Proof Since G_1 has n vertices and G_2 has m vertices, then $G_1 \times G_2$ has mn vertices. Without loss of generality, assume that $m > n$. In the construction of $G_1 \times G_2$, G_2 is drawn n times and then the corresponding adjacency is given according as the adjacency in G_1. Since G_2 is an independent CDPU cycle, from Theorem 3.11, $\sigma_i(G_2) = \frac{m}{2}$. Therefore in $G_1 \times G_2$ there are $\frac{mn}{2}$ vertices in the CDPU set. □

Remark 3.16 In Theorem 3.15, if any one of G_1 or G_2 is C_3, then $|M| = n$, since $\sigma_i(C_3) = 1$.

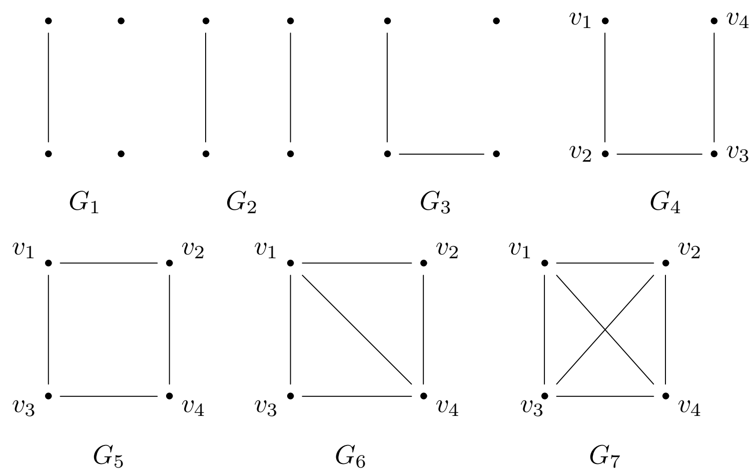

Fig.9: Graphs whose subdivision graphs are bipartite complementary

Theorem 3.17 *The connected graphs, whose subdivision graphs are bipartite complementary are independent CDPU.*

Proof Fig.9 depicts the seven graphs whose subdivision graphs are bipartite self-complementary [2]. In $G_4, M_1 = \{v_1, v_2\}$ gives $f_{M_1}(v_3) = f_{M_1}(v_4) = \{1, 2\}$.

In $G_5, M_2 = \{v_1, v_4\}$ gives $f_{M_2}(v_3) = f_{M_2}(v_2) = \{1\}$.

In $G_6, M_3 = \{v_2, v_3\}$ gives $f_{M_3}(v_1) = f_{M_3}(v_4) = \{1\}$.

In $G_7, M_4 = \{v_1\}$ gives $f_{M_4}(v_2) = f_{M_4}(v_3) = f_{M_4}(v_4) = \{1\}$. Hence M_1, M_2, M_3, M_4 are independent CDPU sets. Thus the connected graphs G_4, G_5, G_6 and G_7 are independent CDPU. □

§4. Conclusion and Scope

As already stated in the introduction, the concept under study has important applications in the field of Chemistry. The study is interesting due to its applications in Computer Networks and Engineering, especially in Control System. In a closed loop control system, signal flow graph representation is used for gain analysis. So in certain control systems specified by certain characteristics, we can find out M, a set consisting of two vertices such that one vertex will be the take off point and other vertex will be the summing point.

Following are some problems that are under investigation:

1. Characterize independent CDPU trees.
2. Characterize unicyclic graphs which are independent CDPU.
3. What is the independent CDPU number for a generalized Peterson graph.
4. What are those classes of graphs with $r(G) = \sigma_i(G)$, where $r(G)$ is the radius of G.

Acknowledgements

The authors are deeply indebted to Professor Dr.B.D.Acharya, SRC-IIIDMS, University of Mysore, Mysore-560 005, India for sparing his valuable time in sharing his many incisive thoughts to propel our vigorous discussion on the content of this paper. They are thankful to the Department of Science & Technology, Government of India for supporting this research under the project No. SR/S4/MS:277/06.

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Math.Combin.Book.Ser. Vol.4 (2009), 75-83

On Smarandachely Harmonic Graphs

D.D.Somashekara and C.R.Veena

Department of Studies in Mathematics, University of Mysore

Manasagangothri MYSORE - 570006, INDIA

E-mail: dsomashekara@yahoo.com, veenacr.maths@gmail.com

Abstract: A graph G is said to be Smarandachely harmonic graph with property P if its vertices can be labeled $1, 2, \cdots, n$ such that the function $f_P : A \to Q$ defined by

$$f_p(H) = \frac{\prod\limits_{v \in V(H)} f(v)}{\sum\limits_{v \in V(H)} f(v)}, \quad H \in A$$

is injective. Particularly, if A is the collection of all paths of length 1 in G (That is, $A = E(G)$), then a Smarandachely harmonic graph is called Strongly harmonic graph. In this paper we show that all cycles, wheels, trees and grids are strongly harmonic graphs. Also we give an upper bound and a lower bound for $\mu(n)$, the maximum number of edges in a strongly harmonic graph of order n.

Key Words: Graph labeling, Smarandachely harmonic graph, strongly harmonic graph.

AMS(2000): 05C78.

§1. Introduction

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. After it was introduced in late 1960's thousands of research articles on graph labelings and their applications have been published.

Recently in 2001, L. W. Beineke and S. M. Hegde [7] introduced the concept of strongly multiplicative graph. A graph with n vertices is said to be Strongly multiplicative if the vertices of G can be labeled with distinct integers $1, 2, \cdots, n$ such that the values on the edge obtained as the product of the labels of their end vertices are all distinct. They have proved that certain classes of graphs are strongly multiplicative. They have also obtained an upper bound for $\lambda(n)$, the maximum number of edges for a given strongly multiplicative graph of order n. In [3], C. Adiga, H.N. Ramaswamy and D. D. Somashekara gave a sharper upper bound for $\lambda(n)$. Further C. Adiga, H. N. Ramaswamy and D. D. Somashekara [1] gave a lower bound for $\lambda(n)$ and proved that the complete bipartite graph $K_{r,r}$ is strongly multiplicative if and only if $r \leq 4$. In 2003, C. Adiga, H. N. Ramaswamy and D. D. Somashekara [2] gave a formula for $\lambda(n)$ and

[1]Received Sept.28, 2009. Accepted Nov. 28, 2009.

also showed that every wheel is strongly multiplicative. Seoud and Zid [9] and Germina and Ajitha [8] have made further contributions to this concept of strongly multiplicative graphs.

In 2000, C. Adiga, and D. D. Somashekara [4] have introduced the concept of Strongly $\star$ - graph and showed that certain classes of graphs are strongly $\star$ - graphs. Also they have obtained a formula, upper and lower bounds for the maximum number of edges in a strongly $\star$ - graph of order n. Baskar Babujee and Vishnupriya [6] have also proved that certain class of graphs are strongly $\star$ - graphs.

A graph with n vertices is said to be Strongly quotient graph if its vertices can be labeled $1, 2, \cdots, n$ so that the values on the edges obtained as the quotient of the labels of their end vertices are all distinct. In [5], C. Adiga, M. Smitha and R. Kaeshgas Zafarani showed that certain class of graphs are strongly quotient graphs. They have also obtained a formula, upper and two different lower bounds for the maximum number of edges in a strongly quotient graph of order n.

In this sequel, we shall introduce the concept of Strongly Harmonic graphs.

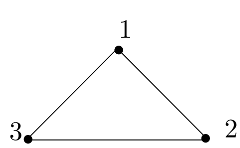
Definition 1.1 *A labeling of a graph G of order n is an injective mapping $f : V(G) \to \{1, 2, \ldots, n\}$.*

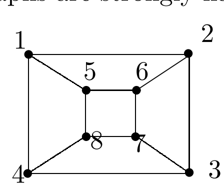
Definition 1.2 *Let G be a graph of order n and A be the set of all paths in G. Then G is said to be Smarandachely harmonic graph with property P if its vertices can be labeled $1, 2, \cdots, n$ such that the function $f_P : A \to Q$ defined by*

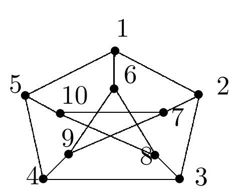
$$f_p(H) = \frac{\prod\limits_{v \in V(H)} f(v)}{\sum\limits_{v \in V(H)} f(v)}, \quad H \in A$$

is injective. In particular if A is the collection of all paths of length 1 in G (That is, $A = E(G)$), then a Smarandachely harmonic graph is called Strongly harmonic graph.

For example, the following graphs are strongly harmonic graphs.







In Section 2, we show that certain class of graphs are strongly harmonic. In Section 3, we give upper and lower bounds for $\mu(n)$, the maximum number of edges in a strongly harmonic graph of order n.

§2. Some Classes of Strongly Harmonic Graphs

Theorem 2.1 *The complete graph K_n is strongly harmonic graph if and only if $n \leq 11$.*

Proof For $n \leq 11$ it is easy to see that K_n are strongly harmonic graphs. When $n = 12$, we have $\frac{4 \cdot 3}{4+3} = \frac{12}{7} = \frac{24}{14} = \frac{12 \cdot 2}{12+2}$. Therefore K_{12} is not strongly harmonic graph and hence any complete graph K_n, for $n \geq 12$ is not strongly harmonic. □

Theorem 2.2 *For all $n \geq 3$, the cycle C_n is strongly harmonic graph.*

Proof Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$ be a cycle of order n. Then consider the following labelling of the graph $v_1 = 1, v_2 = 2, \ldots, v_n = n$. Then the value of the edge $v_k v_{k+1}$ is $\frac{k(k+1)}{2k+1}$, for $1 \leq k < n$. The value of the edge $v_n v_1$ is $\frac{n}{n+1}$. Since $\frac{2}{3} < \frac{n}{n+1} < \frac{6}{5} < \cdots < \frac{n(n-1)}{2n-1}$ for all $n \geq 3$, it follows that the values of the edges are all distinct, proving that every cycle $C_n,\ n \geqslant 3$, is strongly harmonic. □

Theorem 2.3 *Every wheel is strongly harmonic.*

Proof Consider the wheel W_{n+1}, whose rim is the cycle $v_1, v_2, \ldots, v_n, v_1$ and whose hub is the vertex w.

Case (i) $n + 1$ is odd.

Let p be a prime such that $\frac{n}{2} < p < n$. Such a prime p exists by Bertrand's hypothesis. Consider the following labeling of graphs:

$v_1 = 1, v_2 = 2, \cdots, v_{p-1} = p-1, v_p = p+1, \cdots, v_n = n+1, w = p.$

The value of the edge $v_k v_{k+1}$ is $\frac{k(k+1)}{2k+1}$ for $1 \leq k < p-1$ and the value of the edge $v_k v_{k+1}$ is $\frac{(k+1)(k+2)}{2k+3}$ for $p \leq k < n$. The value of the edge $v_{p-1} v_p$ is $\frac{(p-1)(p+1)}{2p}$ and the value of the edge $v_n v_1$ is $\frac{n+1}{n+2}$. Since

$$\frac{2}{3} < \frac{n+1}{n+2} < \frac{6}{5} < \cdots < \frac{(p-2)(p-1)}{2p-3} < \frac{(p-1)(p+1)}{2p} < \frac{(p+1)(p+2)}{2p+3}$$
$$< \cdots < \frac{n(n+1)}{2n+1},$$

the value of the rim edges are all distinct.

The value of the spoke edges are $\frac{p}{p+1}, \frac{2p}{p+2}, \cdots, \frac{(n+1)p}{n+1+p}$. Since $\frac{p}{p+1} < \frac{2p}{p+2} < \cdots < \frac{(p-1)p}{2p-1} < \frac{(p+1)p}{2p+1} < \cdots < \frac{(n+1)p}{n+1+p}$, the value of the spoke edges are all distinct. The

numerator in the values of spoke edges are all divisible by p and the numerator in the values of the rim edges are not divisible by p. Hence the value of the edges of the wheel are all distinct. Hence when $n+1$ is odd, the wheel is strongly harmonic.

Case (ii) $n+1$ is even.

Let p be a prime such that $\frac{n+1}{2} < p < n+1$. Proof follows in the same lines as in case(i). Hence by the choice of p edges of the wheel are all distinct. Therefore wheel is strongly harmonic. □

Theorem 2.4 *Every tree is strongly harmonic graph.*

Proof Label the vertices of the tree using breadth - first search method. To show that the labeling is strongly harmonic it suffices to consider the following two cases.

Case (i) Let $e_1 = (a, b)$ and $e_2 = (a, c)$ be the edges with a common vertex as shown in the Fig.2.1.

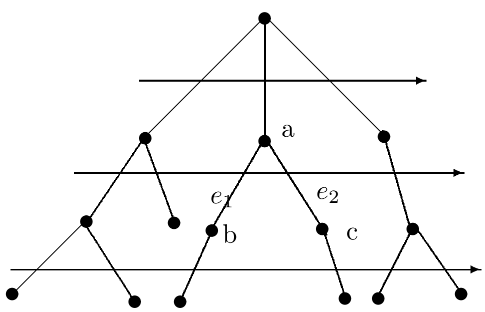


Fig.2.1

From the breadth - first search method of labelling it follows that $a < b < c$. This implies that $\frac{ab}{a+b} < \frac{ac}{a+c}$. Hence the values of the edges with common vertex form a strictly increasing sequence of rational numbers.

Case (ii) Let $e_1 = (a, c)$ and $e_2 = (b, d)$, where the edges e_1 and e_2 fall in the same level as shown in the Fig.2.2 or in two consecutive level as shown in Fig.2.3.

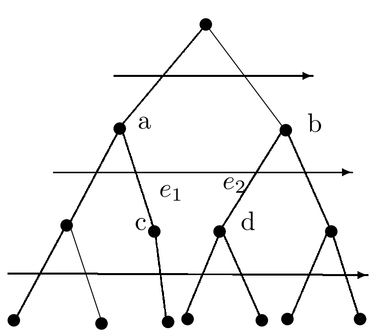


Fig.2.2

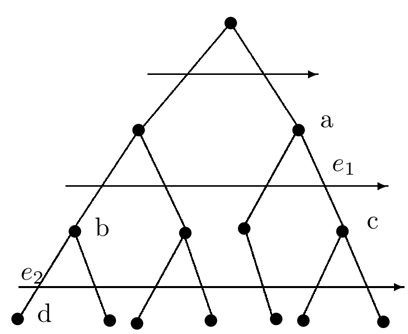


Fig.2.3

From the breadth - first search method of labeling it follows that $a < b < c < d$. This implies that $\frac{ac}{a+c} < \frac{bd}{b+d}$. Hence as indicated by the arrows, the values of the edges form a strictly increasing sequence of rational numbers.

Thus the values of the edges are all distinct. So each tree is strongly harmonic graph. $\square$

Theorem 2.5 *Every grid is strongly harmonic graph.*

Proof Label the vertices of the grid using breadth - first search method. To show that the labeling is strongly harmonic it suffices to consider the following three cases. The first three cases are similar to the two cases considered in the proof of the Theorem 2.4. The last case is when $e_1 = (a, c)$ and $e_2 = (b, c)$ as shown in the Fig.2.4.

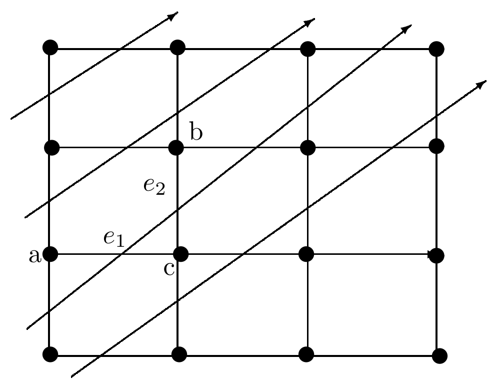


Fig.2.4

In this case from the breadth - first search method of labeling it follows that $a < b < c$ which implies that $\frac{ac}{a+c} < \frac{bc}{b+c}$.

Therefore, as indicated by the arrows, the values of the edges form a strictly increasing sequence of rational numbers. Thus, the values of the edges are all distinct proving that every grid is strongly harmonic. $\square$

§3. Upper and Lower Bounds for $\mu(n)$

In this section we give an upper and a lower bound for $\mu(n)$.

Theorem 3.1 *If $\mu(n)$ denotes the number of edges in a strongly harmonic graph of order n, then*

$$\mu(n) \leq \frac{n(n-1)}{2} - \sum_{k=1}^{2} \left[\frac{\sqrt{4nk + k^2} + k}{4k} \right]$$

$$- \sum_{k=1}^{[\frac{n+12}{48}]} \left[\frac{\sqrt{(4k-1)(4n+4k-1)} + (4k-1)}{16k-4} \right]$$

$$-\sum_{k=1}^{[\frac{n+24}{96}]}\left[\frac{\sqrt{(8k-2)(4n+8k-2)}+(8k-2)}{32k-8}\right]$$

$$+2+\left[\frac{n+12}{48}\right]+\left[\frac{n+24}{96}\right], \tag{1}$$

where $[x]$ *denotes the greatest integer less than or equal to* x.

Proof Given n, the total number of edges in a complete graph of order n is $\frac{n(n-1)}{2}$.

For $7k \le t \le n$, and $t \equiv -k (\bmod\ 4k)$ where $1 \le k \le 2$ the values of the edges e_1 with end vertices $\left(\frac{t+k}{4}, \frac{t^2-k^2}{4k}\right)$ and e_2 with end vertices $\left(\frac{t+k}{2}, \frac{t-k}{2}\right)$ are equal provided $\frac{t^2-k^2}{4k} \le n$ or $t \le \sqrt{4nk+k^2}$. Since $t = 4km - k$, for some positive integer m, we have

$$7k \le 4km - k \le \sqrt{4nk+k^2}.$$

This double inequality yields

$$2 \le m \le \left(\frac{\sqrt{4nk+k^2}+k}{4k}\right).$$

Therefore, the number of such pairs of edges with equal values is

$$\left[\frac{\sqrt{4nk+k^2}+k}{4k}\right]-1. \tag{2}$$

Next for $28k-7 \le t \le n$, and $t \equiv -(4k-1)(\bmod\ (16k-4))$ and $48k-12 \le n$, the values of the edges e_1 with end vertices $\left(\frac{t+(4k-1)}{4}, \frac{t^2-(4k-1)^2}{16k-4}\right)$ and e_2 with end vertices $\left(\frac{t+(4k-1)}{2}, \frac{t-(4k-1)}{2}\right)$ are equal provided $\frac{t^2-(4k-1)^2}{16k-4} \le n$ or $t \le \sqrt{(4k-1)(4n+4k-1)}$. Since $t = (16k-4)m-(4k-1)$, for some positive integer m, we have

$$28k-7 \le (16k-4)m-(4k-1) \le \sqrt{(4k-1)(4n+4k-1)}.$$

This double inequality yields

$$2 \le m \le \left(\frac{\sqrt{(4k-1)(4n+4k-1)}+(4k-1)}{16k-4}\right).$$

Therefore, the number of such pairs of edges with equal values is

$$\left[\frac{\sqrt{(4k-1)(4n+4k-1)}+(4k-1)}{16k-4}\right]-1. \tag{3}$$

For $56k-14 \le t \le n$, and $t \equiv -(8k-2)(\bmod\ (32k-8))$ and $96k-24 \le n$, the values of the edges e_1 with end vertices $\left(\frac{t+(8k-2)}{4}, \frac{t^2-(8k-2)^2}{32k-8}\right)$ and e_2 with end vertices

$\left(\dfrac{t+(8k-2)}{2}, \dfrac{t-(8k-2)}{2}\right)$ are equal and proceeding as above we find that the number of such pairs of edges with equal values is

$$\left[\frac{\sqrt{(4k-1)(2n+4k-1)}+(4k-1)}{16k-4}\right]-1. \tag{4}$$

From equations (2), (3) and (4), we get

$$\mu(n) \le \frac{n(n-1)}{2} - \sum_{k=1}^{2}\left(\left[\frac{\sqrt{4nk+k^2}+k}{4k}\right]-1\right)$$

$$-\sum_{k=1}^{[\frac{n+12}{48}]}\left(\left[\frac{\sqrt{(4k-1)(4n+4k-1)}+(4k-1)}{16k-4}\right]-1\right)$$

$$-\sum_{k=1}^{[\frac{n+24}{96}]}\left(\left[\frac{\sqrt{(4k-1)(2n+4k-1)}+(4k-1)}{16k-4}\right]-1\right)$$

which yields (1). $\square$

Theorem 3.2

$$\mu(n) \ge n + \sum_{k=2}^{n-2} f(k), \quad n \ge 4, \tag{5}$$

where $f(k) = min\left\{n - \left[\dfrac{nk(k-1)}{k(k-1)+n}\right], n-k\right\}$.

Proof Let $A = \left\{\dfrac{rs}{r+s}; 1 \le r < s \le n\right\}$. Then clearly $\mu(n) = |A|$. Consider the array of rational numbers:

$$\begin{array}{cccccc}
\frac{1\cdot 2}{1+2} & \frac{1\cdot 3}{1+3} & \frac{1\cdot 4}{1+4} & \cdots & \frac{1\cdot(n-1)}{1+(n-1)} & \frac{1\cdot n}{1+n} \\[2ex]
 & \frac{2\cdot 3}{2+3} & \frac{2\cdot 4}{2+4} & \cdots & \frac{2\cdot(n-1)}{2+(n-1)} & \frac{2\cdot n}{2+n} \\[2ex]
 & & \frac{3\cdot 4}{3+4} & \cdots & \frac{3\cdot(n-1)}{3+(n-1)} & \frac{3\cdot n}{3+n} \\
 & & & & & \cdots \\
 & & & & & \cdots \\
 & & & & \frac{(n-2)\cdot(n-1)}{(n-2)+(n-1)} & \frac{(n-2)\cdot n}{(n-2)+n} \\[2ex]
 & & & & & \frac{(n-1)\cdot n}{(n-1)+n}
\end{array}$$

Now, let A_1 denote the set of all elements of the first row. Let $A_k,\ 2 \le k \le n-2$ denote the set of all elements of the k-th row which are greater than $\dfrac{(k-1)\cdot n}{(k-1)+n}$ and hence greater than every element of the $(k-1)$-th row. Let $A_{n-1} = \dfrac{(n-1)\cdot n}{(n-1)+n}$. Clearly $A_i \cap A_j = \emptyset$ for $i \ne j$ and $A_i \subset A$, for all $i = 1, 2, \dots, n-1$. Hence

$$\mu(n) = |A| \ge \sum_{i=1}^{n-1} |A_i|. \tag{6}$$

Now one can easily see that

$$\left.\begin{array}{rcl} |A_1| & = & n-1, \\ |A_k| & = & n - max\left\{\left[\dfrac{nk(k-1)}{k(k-1)+n}\right], k\right\}, \quad 2 \le k \le n-2 \\ & = & min\ \left\{n - \left[\dfrac{nk(k-1)}{k(k-1)+n}\right], n-k\right\} = f(k), \\ \text{and} \quad |A_k| & = & 1. \end{array}\right\} \tag{7}$$

Using (7) in (6) we obtain (5). □

The following table gives the values of $\mu(n)$ and upper and lower bounds for $\mu(n)$ found using Theorems 3.1 and 3.2, respectively.

n	$\mu(n)$	Upper bound	Lower bound
4	6	6	6
5	10	10	10
6	15	15	15
7	21	21	21
8	28	28	28
9	36	36	34
10	45	45	41
11	55	55	48
12	64	65	55
13	76	77	63
14	89	90	71
15	102	104	80

n	$\mu(n)$	Upper bound	Lower bound
16	117	119	90
17	133	135	97
18	150	152	107
19	168	170	117
20	183	191	127

Acknowledgments

The authors are thankful to Prof. C. Adiga for proposing the problem and for his valuable suggestions. The authors are also thankful to the Department of Science and Technology, Government of India, New Delhi for the financial support under the grant DST/SR/S4/MS:490/07.

The authors are greatful to the Referee for suggestions which has considerably improved the quality of the paper.

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Math.Combin.Book.Ser. Vol.4 (2009), 84-88

Signed Graph Equation $L^K(S) \sim \overline{S}$

P. Siva Kota Reddy[†] and M. S. Subramanya[††]

[†]Department of Mathematics, Rajeev Institute of Technology,

Industrial Area, B-M Bypass Road, Hassan 573 201, India

[††]Department of Studies in Mathematics, University of Mysore,

Manasagangotri, Mysore 570 006, India

E-mail: reddy_math@yahoo.com, subramanya_ms@rediffmail.com

Abstract: A *Smarandachely k-signed graph (Smarandachely k-marked graph)* is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-singed graph or 2-marked graph is called abbreviated to a *singed graph* or a *marked graph*. We characterize signed graphs S for which $L(S) \sim \overline{S}$, $\overline{S} \sim C_E(S)$ and $L^k(S) \sim \overline{S}$, where $\sim$ denotes switching equivalence and $L(S)$, $\overline{S}$ and $C_E(S)$ are denotes line signed graph, complementary signed Graph and common-edge signed graph of S respectively.

Key Words: Smarandachely k-signed graph, Smarandachely k-marked graph, signed graphs, balance, switching, line signed graph, complementary signed graph, common-edge signed graph.

AMS(2000): 05C22.

§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [7]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A *Smarandachely k-signed graph (Smarandachely k-marked graph)* is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-singed graph or 2-marked graph is called abbreviated to a *singed graph* or a *marked graph*. A signed graph $S = (G, \sigma)$ is *balanced* if every cycle in S has an even number of negative edges (See [8]). Equivalently a signed graph is balanced if product of signs of the edges on every cycle of S is positive.

A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; A signed graph S together with a marking

[1]Received Oct.8, 2009. Accepted Dec. 10, 2009.

μ is denoted by S_μ.

The following characterization of balanced signed graphs is well known.

Proposition 1 (E. Sampathkumar [10]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Behzad and Chartrand [4] introduced the notion of *line signed graph* $L(S)$ of a given signed graph S as follows: $L(S)$ is a signed graph such that $(L(S))^u \cong L(S^u)$ and an edge e_ie_j in $L(S)$ is negative if, and only if, both e_i and e_j are adjacent negative edges in S. Another notion of line signed graph introduced in [6],is as follows: The *line signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$ (see also, E. Sampathkumar et al. [11]. In this paper, we follow the notion of line signed graph defined by M. K. Gill [6].

Proposition 2 *For any signed graph $S = (G, \sigma)$, its line signed graph $L(S) = (L(G), \sigma')$ is balanced.*

Proof We first note that the labeling σ of S can be treated as a marking of vertices of $L(S)$. Then by definition of $L(S)$ we see that $\sigma'(ee') = \sigma(e)\sigma(e')$, for every edge ee' of $L(S)$ and hence, by proposition-1, the result follows. $\square$

Remark: In [2], M. Acharya has proved the above result. The proof given here is different from that given in [2].

For any positive integer k, the k^{th} iterated line signed graph, $L^k(S)$ of S is defined as follows:

$$L^0(S) = S,\ L^k(S) = L(L^{k-1}(S))$$

Corollary *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $L^k(S)$ is balanced.*

Let $S = (G, \sigma)$ be a signed graph. Consider the marking μ on vertices of S defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the signs on the edges incident at v. *Complement* of S is a signed graph $\overline{S} = (\overline{G}, \sigma^c)$, where for any edge $e = uv \in \overline{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S}$ as defined here is a balanced signed graph due to Proposition 1.

The idea of switching a signed graph was introduced by Abelson and Rosenberg [1] in connection with structural analysis of marking μ of a signed graph S. Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further, a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [14]) or *cycle isomorphic* (see [15]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [15]).

Proposition 3 (T. Zaslavasky [15]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. Switching Equivalence of Iterated Line Signed Graphs and Complementary Signed Graphs

In [12], we characterized signed graphs that are switching equivalent to their line signed graphs and iterated line signed graphs. In this paper, we shall solve the equation $L^k(S) \sim \overline{S}$.

We now characterize signed graphs whose complement and line signed graphs are switching equivalent. In the case of graphs the following result is due to Aigner [3] (See also [13] where $H \circ K$ denotes the corona of the graphs H and K [7].

Proposition 4 (M. Aigner [3]) *The line graph $L(G)$ of a graph G is isomorphic with $\overline{G}$ if, and only if, G is either C_5 or $K_3 \circ K_1$.*

Proposition 5 *For any signed graph $S = (G, \sigma)$, $L(S) \sim \overline{S}$ if, and only if, G is either C_5 or $K_3 \circ K_1$.*

Proof Suppose $L(S) \sim \overline{S}$. This implies, $L(G) \cong \overline{G}$ and hence by Proposition-4 we see that the graph G must be isomorphic to either C_5 or $K_3 \circ K_1$.

Conversely, suppose that G is a C_5 or $K_3 \circ K_1$. Then $L(G) \cong \overline{G}$ by Proposition-4. Now, if S any signed graph on any of these graphs, By Proposition-2 and definition of complementary signed graph, $L(S)$ and $\overline{S}$ are balanced and hence, the result follows from Proposition 3. □

In [5], the authors define *path graphs* $P_k(G)$ of a given graph $G = (V, E)$ for any positive integer k as follows: $P_k(G)$ has for its vertex set the set $\mathcal{P}_k(G)$ of all distinct paths in G having k vertices, and two vertices in $\mathcal{P}_k(G)$ are adjacent if they represent two paths $P, Q \in \mathcal{P}_k(G)$ whose union forms either a path P_{k+1} or a cycle C_k in G.

Much earlier, the same observation as above on the formation of a line graph $L(G)$ of a given graph G, Kulli [9] had defined the *common-edge graph* $C_E(G)$ of G as the *intersection graph* of the family $\mathcal{P}_3(G)$ of 2-paths (i.e., paths of length two) each member of which is treated as a set of edges of corresponding 2-path; as shown by him, it is not difficult to see that $C_E(G) \cong L^2(G)$, for any isolate-free graph G, where $L(G) := L^1(G)$ and $L^t(G)$ denotes the t^{th} *iterated line graph* of G for any integer $t \geq 2$.

In [12], we extend the notion of $C_E(G)$ to realm of signed graphs: Given a signed graph $S = (G, \sigma)$ its *common-edge signed graph* $C_E(S) = (C_E(G), \sigma')$ is that signed graph whose underlying graph is $C_E(G)$, the common-edge graph of G, where for any edge (e_1e_2, e_2e_3) in $C_E(S)$, $\sigma'(e_1e_2, e_2e_3) = \sigma(e_1e_2)\sigma(e_2e_3)$.

Proposition 6(E. Sampathkumar et al. [12]) *For any signed graph $S = (G, \sigma)$, its common-edge signed graph $C_E(S)$ is balanced.*

We now characterize signed graph whose complement $\overline{S}$ and common-edge signed graph $C_E(S)$ are switching equivalent. In the case of graphs the following result is due to Simic [13].

Proposition 7(S. K. Simic [13]) *The common-edge graph $C_E(G)$ of a graph G is isomorphic with $\overline{G}$ if, and only if, G is either C_5 or $K_2 \circ \overline{K_2}$.*

Proposition 8 *For any signed graph $S = (G, \sigma)$, $\overline{S} \sim C_E(S)$ if, and only if, G is either C_5 or $K_2 \circ \overline{K_2}$.*

Proof Suppose $\overline{S} \sim C_E(S)$. This implies, $\overline{G} \cong C_E(G)$ and hence by Proposition-7, we see that the graph G must be isomorphic to either C_5 or $K_2 \circ \overline{K_2}$.

Conversely, suppose that G is a C_5 or $K_2 \circ \overline{K_2}$. Then $\overline{G} \cong C_E(G)$ by Proposition-7. Now, if S any signed graph on any of these graphs, By Proposition-6 and definition of complementary signed graph, $C_E(S)$ and $\overline{S}$ are balanced and hence, the result follows from Proposition 3. □

We now characterize signed graphs whose complement and its iterated line signed graphs $L^k(S)$, where $k \geq 3$ are switching equivalent. In the case of graphs the following result is due to Simic [13].

Proposition 9(S. K. Simic [13]) *For any positive integer $k \geq 3$, $L^k(G)$ is isomorphic with $\overline{G}$ if, and only if, G is C_5.*

Proposition 10 *For any signed graph $S = (G, \sigma)$ and for any positive integer $k \geq 3$, $L^k(S) \sim \overline{S}$ if, and only if, G is C_5.*

Proof Suppose $L^k(S) \sim \overline{S}$. This implies, $L^k(G) \cong \overline{G}$ and hence by Proposition-9 we see that the graph G is isomorphic to C_5.

Conversely, suppose that G is isomorphic to C_5. Then $L^k(G) \cong \overline{G}$ by Proposition-9. Now, if S any signed graph on C_5, By Corollary-2.1 and definition of complementary signed graph, $L^k(S)$ and $\overline{S}$ are balanced and hence, the result follows from Proposition 3. □

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Directionally n-signed graphs-II

E. Sampathkumar

Department of Studies in Mathematics, University of Mysore

Manasagangotri, Mysore 570006, India

P. Siva Kota Reddy

Department of Mathematics Rajeev Institute of Technology,

Industrial Area, B-M Bypass Road, Hassan 573 201, India

M. S. Subramanya

Department of Studies in Mathematics, University of Mysore

Manasagangotri, Mysore 570006, India

E-mail: esampathkumar@eth.net; reddy_math@yahoo.com; subramanya_ms@rediffmail.com

Abstract: Let $G = (V, E)$ be a graph. Let V be a vector space of dimensional n. A *Smarandachely labeling* on a graph G is labeling an edge $uv \in E(G)$ by an vector $v \in V$ on (u, v) and $-v$ on (v, u). Then turn the conception *directional labeling* as a special case to Smarandachely labeling. By *directional labeling (or d-labeling)* of an edge $x = uv$ of G by an ordered n-tuple $(a_1, a_2, ..., a_n)$, we mean a labeling of the edge x such that we consider the label on uv as $(a_1, a_2, ..., a_n)$ in the direction from u to v, and the label on x as $(a_n, a_{n-1}, ..., a_1)$ in the direction from v to u. Here, we study graphs, called *(n, d)-sigraphs*, in which every edge is d-labeled by an n-tuple $(a_1, a_2, ..., a_n)$, where $a_k \in \{+, -\}$, for $1 \leq k \leq n$. In this paper, we obtain another characterization of i-balanced (n, d)-sigraphs, introduced the notion of path balance and generalized the notion of local balance in sigraphs to (n, d)-sigraphs. Further, we obtain characterization of path i-balanced (n, d)-sigraphs.

Key Words: Smarandachely labeling, sigraphs, directional labeling, complementation, balance.

AMS(2000): 05C22.

§1. Introduction

For graph theory terminology and notation in this paper we follow the book [1]. All graphs considered here are finite and simple.

Let V be a vector space of dimensional n. A *Smarandachely labeling* on a graph G is labeling an edge $uv \in E(G)$ by an vector $v \in V$ on (u, v) and $-v$ on (v, u). Then turn the

[1]Received Oct.18, 2009. Accepted Dec. 20, 2009.

conception *directional labeling* as a special case to Smarandachely labeling.

There are two ways of labeling the edges of a graph by an ordered n-tuple $(a_1, a_2, ..., a_n)$ (See [7]).

1. *Undirected labeling* or *labeling*. This is a labeling of each edge uv of G by an ordered n-tuple $(a_1, a_2, ..., a_n)$ such that we consider the label on uv as $(a_1, a_2, ..., a_n)$ irrespective of the direction from u to v or v to u.

2. *Directional labeling* or *d-labeling*. This is a labeling of each edge uv of G by an ordered n-tuple $(a_1, a_2, ..., a_n)$ such that we consider the label on uv as $(a_1, a_2, ..., a_n)$ in the direction from u to v, and $(a_n, a_{n-1}, ..., a_1)$ in the direction from v to u.

Note that the d-labeling of edges of G by ordered n-tuples is equivalent to labeling the symmetric digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, where uv is a symmetric arc in $\overrightarrow{G}$ if, and only if, uv is an edge in G, so that if $(a_1, a_2, ..., a_n)$ is the d-label on uv in G, then the labels on the arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are $(a_1, a_2, ..., a_n)$ and $(a_n, a_{n-1}, ..., a_1)$ respectively.

Let H_n be the *n-fold sign group*, $H_n = \{+, -\}^n = \{(a_1, a_2, ..., a_n) : a_1, a_2, ..., a_n \in \{+, -\}\}$ with co-ordinate-wise multiplication. Thus, writing $a = (a_1, a_2, ..., a_n)$ and $t = (t_1, t_2, ..., t_n)$ then $at := (a_1t_1, a_2t_2, ..., a_nt_n)$. For any $t \in H_n$, the *action* of t on H_n is $a^t = at$, the co-ordinate-wise product.

Let $n \geq 1$ be a positive integer. An *n-sigraph (n-sidigraph)* is a graph $G = (V, E)$ in which each edge (arc) is labeled by an ordered n-tuple of signs, i.e., an element of H_n. A *sigraph* $G = (V, E)$ is a graph in which each edge is labeled by $+$ or $-$. Thus a 1-sigraph is a sigraph. Sigraphs are well studied in literature (See for example [2]-[4], [8]-[9]). In this paper, we study graphs in which each edge is labeled by an ordered n-tuple $a = (a_1, a_2, ..., a_n)$ of signs (i.e, an element of H_n) in one direction but in the other direction its label is the reverse: $a^r = (a_n, a_{n-1}, ..., a_1)$, called *directionally labeled n-signed graphs* (or *(n, d)-sigraphs*).

Note that an n-sigraph $G = (V, E)$ can be considered as a symmetric digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, where both $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are arcs if, and only if, uv is an edge in G. Further, if an edge uv in G is labeled by the n-tuple $(a_1, a_2, ..., a_n)$, then in $\overrightarrow{G}$ both the arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are labeled by the n-tuple $(a_1, a_2, ..., a_n)$.

In [5,6], we have initiated a study of $(3, d)$ and $(4, d)$-Sigraphs. Also, we discuss some applications of $(3, d)$ and $(4, d)$-Sigraphs in real life situations.

In [7], we introduce the notion of complementation and generalize the notion of balance in sigraphs to the directionally n-signed graphs. We look at two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge. Also we gave some motivation to study (n, d)-sigraphs in connection with relations among human beings in society.

In this paper, we introduce the notion of path balance and we generalize the notion of local balance in sigraphs (a graph whose edges have signs) to the more general context of graphs with multiple signs on their edges.

In [7], we define complementation and isomorphism for (n, d)-sigraphs as follows: For any $t \in H_n$, the *t-complement* of $a = (a_1, a_2, ..., a_n)$ is: $a^t = at$. The *reversal* of $a = (a_1, a_2, ..., a_n)$ is: $a^r = (a_n, a_{n-1}, ..., a_1)$. For any $T \subseteq H_n$, and $t \in H_n$, the *t-complement* of T is $T^t = \{a^t : a \in T\}$.

For any $t \in H_n$, the *t-complement* of an (n, d)-sigraph $G = (V, E)$, written G^t, is the same graph but with each edge label $a = (a_1, a_2, ..., a_n)$ replaced by a^t. The *reversal* G^r is the same graph but with each edge label $a = (a_1, a_2, ..., a_n)$ replaced by a^r.

Let $G = (V, E)$ and $G' = (V', E')$ be two (n, d)-sigraphs. Then G is said to be *isomorphic* to G' and we write $G \cong G'$, if there exists a bijection $\phi : V \to V'$ such that if uv is an edge in G which is d-labeled by $a = (a_1, a_2, ..., a_n)$, then $\phi(u)\phi(v)$ is an edge in G' which is d-labeled by a, and conversely.

For each $t \in H_n$, an (n, d)-sigraph $G = (V, E)$ is *t-self complementary*, if $G \cong G^t$. Further, G is *self reverse*, if $G \cong G^r$.

Proposition 1(E. Sampathkumar et al. [7]) *For all $t \in H_n$, an (n, d)-sigraph $G = (V, E)$ is t-self complementary if, and only if, G^a is t-self complementary, for any $a \in H_n$.*

Let $v_1, v_2, ..., v_m$ be a cycle C in G and $(a_{k1}, a_{k2}, ..., a_{kn})$ be the n-tuple on the edge $v_k v_{k+1}, 1 \leq k \leq m - 1$, and $(a_{m1}, a_{m2}, ..., a_{mn})$ be the n-tuple on the edge $v_m v_1$.

For any cycle C in G, let $\mathcal{P}(\overrightarrow{C})$ denotes the product of the n-tuples on C given by $(a_{11}, a_{12}, ..., a_{1n})(a_{21}, a_{22}, ..., a_{2n})...(a_{m1}, a_{m2}, ..., a_{mn})$ and

$$\mathcal{P}(\overleftarrow{C}) = (a_{mn}, a_{m(n-1)}, ..., a_{m1})(a_{(m-1)n}, a_{(m-1)(n-1)}, ..., a_{(m-1)1})...(a_{1n}, a_{1(n-1)}, ..., a_{11}).$$

Similarly, for any path P in G, $\mathcal{P}(\overrightarrow{P})$ denotes the product of the n-tuples on P given by

$$(a_{11}, a_{12}, ..., a_{1n})(a_{21}, a_{22}, ..., a_{2n})...(a_{m-1,1}, a_{m-1,2}, ..., a_{m-1,n})$$

and

$$\mathcal{P}(\overleftarrow{P}) = (a_{(m-1)n}, a_{(m-1)(n-1)}, ..., a_{(m-1)1})...(a_{1n}, a_{1(n-1)}, ..., a_{11}).$$

An n-tuple $(a_1, a_2, ..., a_n)$ is *identity n-tuple*, if each $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple.* Further an n-tuple $a = (a_1, a_2, ..., a_n)$ is *symmetric*, if $a^r = a$, otherwise it is a *non-symmetric n-tuple.* In (n, d)-sigraph $G = (V, E)$ an edge labeled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Note that the above products $\mathcal{P}(\overrightarrow{C})$ $(\mathcal{P}(\overrightarrow{P}))$ as well as $\mathcal{P}(\overleftarrow{C})$ $(\mathcal{P}(\overleftarrow{P}))$ are n-tuples. In general, these two products need not be equal. However, the following holds.

Proposition 2 *For any cycle C (path P) of an (n, d)-sigraph $G = (V, E)$, $\mathcal{P}(\overleftarrow{C}) = \mathcal{P}(\overrightarrow{C})^r$ $(\mathcal{P}(\overleftarrow{P})$ $= \mathcal{P}(\overrightarrow{P})^r)$.*

Proof By the definition, we have

$$
\begin{aligned}
&\mathcal{P}(\overleftarrow{C})^r\\
&= ((a_{mn}, a_{m(n-1)}, ..., a_{m1})(a_{(m-1)n}, a_{(m-1)(n-1)}, ..., a_{(m-1)1})...(a_{1n}, a_{1(n-1)}, ..., a_{11}))^r\\
&= ((a_{m1}, a_{m2}, ...a_{mn})^r(a_{(m-1)1}, a_{(m-1)2}, ...a_{(m-1)n})^r...(a_{11}, a_{12}, ...a_{1n})^r)^r\\
&= ((a_{m1}, a_{m2}, ...a_{mn})(a_{(m-1)1}, a_{(m-1)2}, ...a_{(m-1)n})...(a_{11}, a_{12}, ...a_{1n}))\\
&= \mathcal{P}(\overrightarrow{C}).
\end{aligned}
$$

Similarly, we can prove $\mathcal{P}(\overleftarrow{P}) = \mathcal{P}(\overrightarrow{P})^r$. □

Corollary 2.1 *For any cycle C (path P), $\mathcal{P}(\overleftarrow{C}) = \mathcal{P}(\overrightarrow{C})$ ($\mathcal{P}(\overleftarrow{P}) = \mathcal{P}(\overrightarrow{P})$) if, and only if, $\mathcal{P}(\overrightarrow{C})$ ($\mathcal{P}(\overrightarrow{P})$) is a symmetric n-tuple. Furthermore, $\mathcal{P}(\overrightarrow{C})$ ($\mathcal{P}(\overrightarrow{P})$) is the identity n-tuple if, and only if, $\mathcal{P}(\overleftarrow{C})$ ($\mathcal{P}(\overleftarrow{P})$) is.*

For any subset Y of $\overrightarrow{E}(G) = \{(u,v) :\ uv \textit{ is an edge in } G\}$, the set of all arcs in G, the product of the set Y is the product of the n-tuples of its arcs and it is denoted by $\mathcal{P}(Y)$. If Y_1 and Y_2 are disjoint sets, the product of the union of Y_1 and Y_2 is the product of the n-tuples of the two sets:

$$\mathcal{P}(Y_1 \cup Y_2) = \mathcal{P}(Y_1).\mathcal{P}(Y_2).$$

The following Proposition gives a similar result about the symmetric difference of two sets of arcs.

Proposition 3 *If Y_1 and Y_2 are two subsets of $\overrightarrow{E}(G)$ of an (n,d)-sigraph $G = (V,E)$, then $\mathcal{P}(Y_1 \oplus Y_2) = \mathcal{P}(Y_1).\mathcal{P}(Y_2)$.*

Proof We know that

$$Y_1 = (Y_1 - Y_2) \cup (Y_1 \cap Y_2) \text{ and } Y_2 = (Y_2 - Y_1) \cup (Y_1 \cap Y_2).$$

Since each of these is a union of disjoin sets, we have

$$\mathcal{P}(Y_1) = \mathcal{P}(Y_1 - Y_2).\mathcal{P}(Y_1 \cap Y_2) \text{ and } \mathcal{P}(Y_2) = \mathcal{P}(Y_2 - Y_1).\mathcal{P}(Y_1 \cap Y_2).$$

Multiplying these equations we get that

$$\mathcal{P}(Y_1).\mathcal{P}(Y_2) = \mathcal{P}(Y_1 - Y_2).\mathcal{P}(Y_2 - Y_1).\mathcal{P}(Y_1 \cap Y_2).\mathcal{P}(Y_1 \cap Y_2).$$

Since $\mathcal{P}(Y_1 \cap Y_2).\mathcal{P}(Y_1 \cap Y_2)$ is always identity n-tuple, and since $Y_1 - Y_2$ and $Y_2 - Y_1$ are disjoint,

$$\mathcal{P}(Y_1).\mathcal{P}(Y_2) = \mathcal{P}[(Y_1 - Y_2) \cup (Y_2 - Y_1)].$$

Thus, $\mathcal{P}(Y_1).\mathcal{P}(Y_2) = \mathcal{P}(Y_1 \oplus Y_2)$. □

Corollary 3.1 *Two sets of edges Y_1 and Y_2 have the same n-tuple if, and only if, their symmetric difference $Y_1 \oplus Y_2$ is identity.*

§2. Balance in an (n, d)-sigraph

In [7], we defined two notions of balance in an (n, d)-sigraph $G = (V, E)$ as follows.

Definition. *Let $G = (V, E)$ be an (n, d)-sigraph. Then,*

(i) G is identity balanced *(or* i-balanced*), if $P(\overrightarrow{C})$ on each cycle of G is the identity n-tuple, and*

(ii) G is balanced*, if every cycle contains an even number of non-identity edges.*

Note An i-balanced (n, d)-sigraph need not be balanced and conversely. For example, consider the $(4, d)$-sigraphs in Fig.1. In Fig.1(a) G is an i-balanced but not balanced, and in Fig.1(b) G is balanced but not i-balanced.

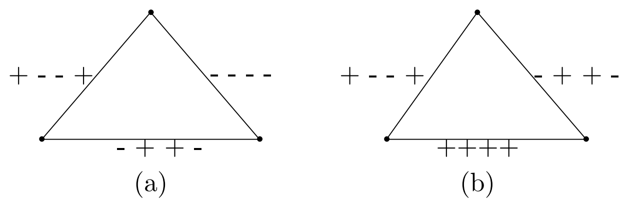

Fig.1

2.1 Criteria for Balance

An (n, d)-sigraph $G = (V, E)$ is i-balanced if each non-identity n-tuple appears an even number of times in $P(\overrightarrow{C})$ on any cycle of G.

However, the converse is not true. For example see Fig.2(a). In Fig.2(b), the number of non-identity 4-tuples is even and hence it is balanced. But it is not i-balanced, since the 4-tuple $(+ + - -)$ (as well as $(- - + +)$) does not appear an even number of times in $P(\overrightarrow{C})$ of 4-tuples.

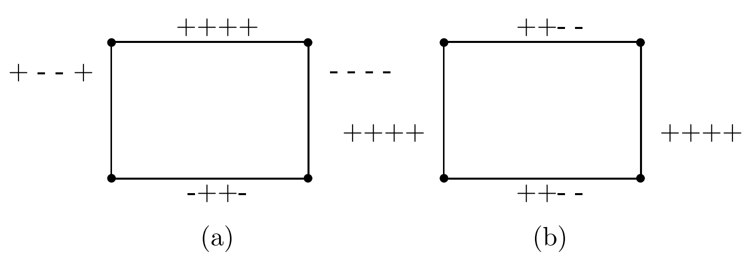

Fig.2

In [7], we obtained following characterizations of balanced and i-balanced (n, d)-sigraphs.

Proposition 4(E. Sampathkumar et al. [7]) *An (n, d)-sigraph $G = (V, E)$ is balanced if, and only if, there exists a partition $V_1 \cup V_2$ of V such that each identity edge joins two vertices in*

V_1 or V_2, and each non-identity edge joins a vertex of V_1 and a vertex of V_2.

As earlier we defined, let $P(C)$ denote the product of the n-tuples in $P(\overrightarrow{C})$ on any cycle C in an (n,d)-sigraph $G = (V, E)$.

Proposition 5(E. Sampathkumar et al. [7]) *An (n,d)-sigraph $G = (V, E)$ is i-balanced if, and only if, for each k, $1 \leq k \leq n$, the number of n-tuples in $P(C)$ whose k^{th} co-ordinate is $-$ is even.*

In H_n, let S_1 denote the set of non-identity symmetric n-tuples and S_2 denote the set of non-symmetric n-tuples. The product of all n-tuples in each $S_k, 1 \leq k \leq 2$ is the identity n-tuple.

Proposition 6(E. Sampathkumar et al. [7]) *An (n,d)-sigraph $G = (V, E)$ is i-balanced, if both of the following hold:*

(i) In $P(C)$, each n-tuple in S_1 occurs an even number of times, or each n-tuple in S_1 occurs odd number of times (the same parity, or equal mod 2).

(ii) In $P(C)$, each n-tuple in S_2 occurs an even number of times, or each n-tuple in S_2 occurs an odd number of times.

In this paper, we obtained another characterization of i-balanced (n,d)-sigraphs as follows:

Proposition 7 *An (n,d)-sigraph $G = (V, E)$ is i-balanced if, and only if, any two vertices u and v have the property that for any two edge distinct $u-v$ paths $\overrightarrow{P_1} = (u = u_0, u_1, ..., u_m = v$ and $\overrightarrow{P_2} = (u = v_0, v_1, ..., v_n = v)$ in G, $\mathcal{P}(\overrightarrow{P_1}) = (\mathcal{P}(\overrightarrow{P_2}))^r$ and $\mathcal{P}(\overrightarrow{P_2}) = (\mathcal{P}(\overrightarrow{P_1}))^r$.*

Proof Suppose that G is i-balanced. The paths $\overrightarrow{P_1}$ and $\overrightarrow{P_2}$ may be combined to form is either a cycle or union of cycles. That is, $P_1 \cup P_2 = (u = u_0, u_1, ..., u_m = v = v_n, v_{n-1}, ..., v_0 = u)$. Since $\mathcal{P}(\overrightarrow{P_1} \cup \overleftarrow{P_2})$=identity n-tuple e.

$$\mathcal{P}(\overrightarrow{P_1}).\ \mathcal{P}(\overleftarrow{P_2}) = e, \quad \mathcal{P}(\overrightarrow{P_1}) = \mathcal{P}(\overleftarrow{P_2}) = (\mathcal{P}(\overrightarrow{P_2}))^r.$$

The converse is obvious. □

Corollary 7.1 *In an i-balanced (n,d)-sigraph G if two vertices are joined by at least 3 paths then the product of n tuples on any paths joining them must be symmetric.*

A graph $G = (V, E)$ is said to be k-connected for some positive integer k, if between any two vertices there exists at least k disjoint paths joining them.

Corollary 7.2 *If the underlying graph of an i-balanced (n,d)-sigraph is 3-connected, then all the edges in G must be labeled by a symmetric n-tuple.*

Corollary 7.3 *A complete (n,d)-sigraph on $p \geq 4$ is i-balanced then all the edges must be labeled by symmetric n-tuple.*

2.2 Complete (n, d)-sigraphs

An (n, d)-sigraph is *complete*, if its underlying graph is complete.

Proposition 8 *The four triangles constructed on four vertices $\{a, b, c, d\}$ can be directed so that given any pair of vertices say (a, b) the product of the edges of these 4 directed triangles is the product of the n-tuples on the arcs $\overrightarrow{ab}$ and $\overrightarrow{ba}$*

Proof The four triangles constructed on these vertices are (abc), (adb), (cad), (bcd). Consider the 4 directed triangles $(\overrightarrow{abc})$, $(\overrightarrow{adb})$, $(\overrightarrow{cad})$, $(\overrightarrow{bcd})$ for the pair ab. Then

$$\begin{aligned}\mathcal{P} &= \mathcal{P}(\overrightarrow{abc}).\mathcal{P}(\overrightarrow{adb}).\mathcal{P}(\overrightarrow{acd}).\mathcal{P}(\overrightarrow{bcd}) \\ &= [\mathcal{P}(\overrightarrow{ab}).\mathcal{P}(\overrightarrow{ca}).\mathcal{P}(\overrightarrow{bc})].\ [\mathcal{P}(\overrightarrow{ad}).\mathcal{P}(\overrightarrow{db}).\mathcal{P}(\overrightarrow{ba})] \\ &\quad [\mathcal{P}(\overrightarrow{ca}).\mathcal{P}(\overrightarrow{ad}).\mathcal{P}(\overrightarrow{cd})][\mathcal{P}(\overrightarrow{bc}).\mathcal{P}(\overrightarrow{db}).\mathcal{P}(\overrightarrow{cd})] \\ &= [\mathcal{P}(\overrightarrow{ab}).\mathcal{P}(\overrightarrow{ba})].\ [\mathcal{P}(\overrightarrow{ca}).\mathcal{P}(\overrightarrow{ca})].\ [\mathcal{P}(\overrightarrow{bc}).\mathcal{P}(\overrightarrow{bc})] \\ &\quad [\mathcal{P}(\overrightarrow{ad}).\mathcal{P}(\overrightarrow{ad})].\ [\mathcal{P}(\overrightarrow{db}).\mathcal{P}(\overrightarrow{db})].\ [\mathcal{P}(\overrightarrow{cd}).\mathcal{P}(\overrightarrow{cd})] \\ &= \mathcal{P}(\overrightarrow{ab})\mathcal{P}(\overrightarrow{ba})\end{aligned}$$

□

Corollary 8.1 *The product of the n-tuples of the four triangles constructed on four vertices $\{a, b, c, d\}$ is identity if at least one edge is labeled by a symmetric n-tuple.*

The *i-balance base* with axis a of a complete (n, d)-sigraph $G = (V, E)$ consists list of the product of the n-tuples on the triangles containing a.

Proposition 9 *If the i-balance base with axis a and n-tuple of an edge adjacent to a is known, the product of the n-tuples on all the triangles of G can be deduced from it.*

Proof Given a base with axis a and the n-tuple of the arc $\overrightarrow{ab}$ be $(a_1, a_2, \cdots, a_n)$.. Consider a triangle (bcd) whose n-tuple is not given by the base. Let $\mathcal{P}' = \mathcal{P}(\overrightarrow{abc}).\mathcal{P}(\overrightarrow{adb}).\mathcal{P}(\overrightarrow{acd})$. Hence, $\mathcal{P}'$ is known from the base with axis a. Let $\mathcal{P}$ be defined as in Proposition-8; we then have $\mathcal{P} = \mathcal{P}'.\ \mathcal{P}(\overrightarrow{bcd})$. By Proposition-8, $\mathcal{P} = \mathcal{P}(\overrightarrow{ab}).\mathcal{P}(\overrightarrow{ba})$. Thus, $\mathcal{P}(\overrightarrow{bcd}) = \mathcal{P}'.\mathcal{P}(\overrightarrow{ab}).\mathcal{P}(\overrightarrow{ba})$. □

Remark 10 In the statement of above Proposition, it is not necessary to know the n-tuple of an edge incident at a. But it is sufficient that an edge incident at a is a symmetric n-tuple.

Proposition 11 *A complete (n, d)-sigraph $G = (V, E)$ is i-balanced if, and only if, all the triangles of a base are identity.*

Proof If all the triangles of a base are identity, all the triangles of the (n, d)-sigraph are identity. Indeed, for any triangle (bed) not appearing in the base with axis a, we have

$$\mathcal{P}(\overrightarrow{bcd}) = \mathcal{P}(\overrightarrow{abc}).\ \mathcal{P}(\overrightarrow{abd}).\ \mathcal{P}(\overrightarrow{acd})\text{=identity}.$$

Conversely, if the (n, d)-sigraph is i-balanced, all these triangles are identity and particular those of a base. □

Corollary 11.1 *All the triangles of a complete (n,d)-sigraph $G=(V,E)$ are i-unbalanced if, and only if, all the triangles of a base are non-identity.*

Proposition 12 *The number of i-balanced complete (n,d)-sigraphs of m vertices is p^{m-1}, where $p=2^{\lceil n/2\rceil}$.*

Proof In a graph G of m vertices, there are $(m-1)$ edges containing a; each of these edges has $p=2^{\lceil n/2\rceil}$ possibilities,since each edge must be labelled by an symmetric n-tuple, by Corollary-7.3. Hence in all, p^{m-1} possibilities, where $p=2^{\lceil n/2\rceil}$. Starting from each of these possibilities, a base with axis a can be constructed, of which all the triangles are identity. □

§3. Path Balance in an (n,d)-sigraph

Definition *Let $G=(V,E)$ be an (n,d)-sigraph. Then G is*

1. Path i-balanced, if any two vertices u and v satisfy the property that for any $u-v$ paths P_1 and P_2 from u to v, $\mathcal{P}(\overrightarrow{P}_1)=\mathcal{P}(\overrightarrow{P}_2)$.

2. Path balanced *if any two vertices u and v satisfy the property that for any $u-v$ paths P_1 and P_2 from u to v have same number of non identity n-tuples.*

Clearly, the notion of path balance and balance coincides. That is an (n,d)-sigraph is balanced if, and only if, G is path balanced.

If an (n,d) sigraph G is i-balanced then G need not be path i-balanced and conversely.

The following result gives a characterization path i-balanced (n,d)-sigraphs.

Theorem 13 *An (n,d)-sigraph is path i-balanced if, and only if, any two vertices u and v satisfy the property that for any two vertex disjoint $u-v$ paths P_1 and P_2 from u to v, $\mathcal{P}(\overrightarrow{P}_1)=\mathcal{P}(\overrightarrow{P}_2)$.*

Proof Necessary: Suppose that G is path i-balanced. Then clearly for any two vertex disjoint paths P_1 and P_2 from one vertex to another, $\mathcal{P}(\overrightarrow{P}_1)=\mathcal{P}(\overrightarrow{P}_2)$.

Sufficiency: Suppose that for any two vertex disjoint paths P_1 and P_2 from one vertex to another, $\mathcal{P}(\overrightarrow{P}_1)=\mathcal{P}(\overrightarrow{P}_2)$ and that G is not path i-balanced. Let $S=\{(u,v):$ there exists paths P and Q from u to v with $\mathcal{P}(\overrightarrow{P})\neq\mathcal{P}(\overrightarrow{Q})\}$. Let $(u,v)\in S$ such that there exists paths P_1 and P_2 such that P_1 has length $d(u,v)$. Then by the hypothesis, the paths P_1 and P_2 must have a common point say w Let P_3 and P_4 be the subpaths from u to w and P_5 and P_6 be the subpaths from w to v. Now either $(u,w)\in S$ or $(w,v)\in S$. This gives a contradiction to the choice of u and v. This completes the proof. □

§4. Local Balance in an (n,d)-Signed Graph

The notion of local balance in signed graph was introduced by F. Harary [3]. A signed graph $S=(G,\sigma)$ is locally at a vertex v, or S is *balanced at* v, if all cycles containing v are balanced. A cut point in a connected graph G is a vertex whose removal results in a disconnected graph.

The following result due to Harary [3] gives interdependence of local balance and cut vertex of a signed graph.

Theorem 14(F. Harary [3]) *If a connected signed graph $S = (G, \sigma)$ is balanced at a vertex u. Let v be a vertex on a cycle C passing through u which is not a cut point, then S is balanced at v.*

We now extend the notion of local balance in signed graph to (n, d)-signed graphs.

Definition *Let $G = (V, E)$ be a (n, d)-sigraph. Then for any vertices $v \in V(G)$, G is* locally i-balanced at v *(*locally balanced at v*) if all cycles in G containing v is i-balanced (balanced).*

Analogous to the theorem we have the following for an (n, d) sigraph.

Theorem 15 *If a connected (n, d)-signed graph $G = (V, E)$ is locally i-balanced (locally balanced) at a vertex u and v be a vertex on a cycle C passing through u which is not a cut point, then S is locally i-balanced(locally balanced) at v.*

Proof Suppose that G is i-balanced at u and v be a vertex on a cycle C passing through u which is not a cut point. Assume that G is not i-balanced at v. Then there exists a cycle C_1 in G which is not i-balanced. Since G is balanced at u, the cycle C is i-balanced.

With out loss of generality we may assume that $u \notin C$ for if u is in C, then $\mathcal{P}(C)$ is identity, since G is i-balanced at u. Let $e = uw$ be an edge in C. Since v is not a cut point there exists a cycle C_0 containing e and v. Then C_0 consists of two paths P_1 and P_2 joining u and v.

Let v_1 be the first vertex in P_1 and v_2 be a vertex in P_2 such that $v_1 \neq v_2 \in C$, such points do exist since v is not a cut point and $v \in C$. Since $u, v \in C_0$. Let P_3 be the path on C_0 from v_1 and v_2, P_4 be a path in C containing v and P_5 is the path from v_1 to v_2. Then $P_5 \cup P_4$ and $P_3 \cup P_5$ are cycles containing u and hence are i-balanced, since they contain u. That is $\mathcal{P}(P_3) = (\mathcal{P}(P_5))^r$ so that $C = P_3 \cup P_5$ is i-balanced. This completes the proof. By using the same arguments we can prove the result for local balance. □

Acknowledgement

The authors are thankful to Department of Science and Technology, Government of India, New Delhi for the financial support under the project grant SR/S4/MS:275/05.

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Math.Combin.Book.Ser. Vol.4 (2009), 99-109

Genus Distribution for a Graph

Liangxia Wan

(Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China)

Hong-Jian Lai

(Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA)

Yanpei Liu

(Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China)

E-mail: wanliangxia@126.com

Abstract: In this paper we develop the technique of a distribution decomposition for a graph. A formula is given to determine genus distribution of a cubic graph. Given any connected graph, it is proved that its genus distribution is the sum of those for some cubic graphs by using the technique.

Key Words: Joint tree; genus distribution; embedding distribution; Smarandachely k-drawing.

AMS(2000): 05C10.

§1. Introduction

We consider finite connected graphs. Surfaces are orientable 2-dimensional compact manifolds without boundaries. Embeddings of a graph considered are always assumed to be orientable 2-cell embeddings. Given a graph G and a surface S, a *Smarandachely k-drawing* of G on S is a homeomorphism ϕ: $G \rightarrow S$ such that $\phi(G)$ on S has exactly k intersections in $\phi(E(G))$ for an integer k. If $k = 0$, i.e., there are no intersections between in $\phi(E(G))$, or in another words, each connected component of $S - \phi(G)$ is homeomorphic to an open disc, then G has an 2-cell embedding on S. If G can be embedded on surfaces S_r and S_t with genus r and t respectively, then it is shown in [1] that for any k with $r \leqslant k \leqslant t$, G has an embedding on S_k. Naturally, the *genus of a graph* is defined to be the minimum genus of a surface on which the graph can be embedded. Given a graph, *how many distinct embeddings does it have on each surface*? This is the genus distribution problem, first investigated by Gross and Furst [4]. As determining the genus of a graph is NP-complete [15], it appears more difficult and significant to determine the genus distribution of a graph.

[1]Partially supported by NNSFC under Grants No. 10871021

[2]Received Oct.20, 2009. Accepted Dec. 25, 2009.

There have been results on genus distribution for some particular types of graphs (see [3], [5], [8], [9], [11]-[17], among others). In [6], Liu discovered the joint trees of a graph which provide a substantial foundation for us to solve the genus distribution of a graph. For a given embedding G_σ of a graph G, one can find the surface, embedding surface or associate surface, which G_σ embeds on by applying the associated joint tree. In fact, genus distribution of G is that of the set of all of its embedding surfaces. This paper first study genus distributions of some sets of surfaces and then investigate the genus distribution of a generic graph by using the surface sorting method developed in [16].

Preliminaries will be briefed in the next section. In Section 3, surfaces Q_j^i will be introduced. We shall investigate the genus distribution of surface sets Q_j^0 and Q_j^1 for $1 \leqslant j \leqslant 24$, and derive the related recursive formulas. In Section 4, a recursion formula of the genus distribution for a cubic graph is given. In the last section, we show that the genus distribution of a general graph can be transformed into genus distribution of some cubic graphs by using a technique we develop in this paper.

§2. Preliminaries

For a graph G, a *rotation at a vertex* v is a cyclic permutation of edges incident with v. A *rotation system* of G is obtained by giving each vertex of G a rotation. Let ρ_v denote the valence of vertex v which is the number of edges incident with v. The number of rotations systems of G is $\prod\limits_{v\in V(G)}(\rho_v - 1)!$. Edmonds found that there is a bijection between the rotations systems of a graph and its embeddings [2]. Youngs provided the first proof published [18]. Thus, the number of embeddings of G is $\prod\limits_{v\in V(G)}(\rho_v - 1)!$. Let $g_i(G)$ denote the number of embeddings of G with the genus i $(i \geq 0)$. Then, the *genus distribution* of G is the sequence $g_0(G), g_1(G), g_2(G), \cdots$. The *genus polynomial* of G is $f_G(x) = \sum\limits_{i\geq 0} g_i(G)x^i$.

Given a spanning tree T of G, the joint trees of G are obtained by splitting each non-tree edge e into two semi-edges e and e^-. Given a rotation system σ of G, G_σ, $\widetilde{T}_\sigma$ and $\mathcal{P}^\sigma_{\widetilde{T}}$ denote the associated embedding, joint tree and embedding surface which G_σ embedded on respectively. There is a bijection btween embeddings and joint trees of G such that G_σ corresponds to $\widetilde{T}_\sigma$. Given a joint tree $\widetilde{T}$, a *sub-joint tree* $\widetilde{T}_1$ of $\widetilde{T}$ is a graph consisting of T_1 and semi-edges incident with vertices of T_1 where T_1 is a tree and $V(T_1) \subseteq V(T)$. A sub-joint tree $\widetilde{T}_1$ of $\widetilde{T}$ is called *maximal* if there is not a tree T_2 such that $V(T_1) \subset V(T_2) \subseteq V(T)$.

A *linear sequence* $S = abc\cdots z$ is a sequence of letters satisfying with a relation $a \prec b \prec c \prec \cdots \prec z$. Given two linear sequences S_1 and S_2, the *difference sequence* S_1/S_2 is obtained by deleting letters of S_2 in S_1. Since a surface is obtained by identifying a letter with its inverse letter on a special polygon along the direction, a surface is regarded as that polygon such that a and a^- occur only once for each $a \in S$ in this sense.

Let $\mathcal{S}$ be the collection of surfaces. Let $\gamma(S)$ be the genus of a surface S. In order to determine $\gamma(S)$, an equivalence is defined by Op1, Op2 and Op3 on $\mathcal{S}$ as follows:

Op 1. $AB \sim (Ae)(e^-B)$ where $e \notin AB$;

Op 2. $Ae_1e_2Be_2^-e_1^- \sim AeBe^- = Ae^-Be$ where $e \notin AB$;
Op 3. $Aee^-B \sim AB$ where $AB \neq \emptyset$

where AB is a surface.

Thus, S is equivalent to one, and only one of the canonical forms of surfaces $a_0a_0^-$ and $\prod_{k=1}^{i} a_kb_ka_k^-b_k^-$ which are the sphere and orientable surfaces of genus $i(i \geq 1)$.

Lemma 2.1 ([6]) *Let A and B be surfaces. If $a, b \notin B$, and if $A \sim Baba^-b^-$, then $\gamma(A) = \gamma(B) + 1$.*

Lemma 2.2 ([7]) *Let A, B, C, D and E be linear sequences and let $ABCDE$ be a surface. If $a, b \notin ABCDE$, then $AaBbCa^-Db^-E \sim ADCBEaba^-b^-$.*

Lemma 2.3 ([13],[16]) *Let A, B, C and D be linear sequences and let $ABCD$ be a surface. If $a \neq b \neq c \neq a^- \neq b^- \neq c^-$ and if $a, b, c \notin ABCD$, then each of the following holds.*

(i) $aABa^-CD \sim aBAa^-CD \sim aABa^-DC$.
(ii) $AaBa^-bCb^-cDc^- \sim aBa^-AbCb^-cDc^- \sim aBa^-bCb^-AcDc^-$.
(iii) $AaBa^-bCb^-cDc^- \sim BaAa^-bCb^-cDc^- \sim CaAa^-bBb^-cDc^- \sim DaAa^-bBb^-cCc^-$.

For a set of surfaces M, let $g_i(M)$ denote the number of surfaces with the genus i in M. Then, the *genus distribution* of M is the sequence $g_0(M), g_1(M), g_2(M), \cdots$. The *genus polynomial* is $f_M(x) = \sum_{i \geq 0} g_i(M)x^i$.

§3. Genus Distribution for Q_j^1

Let $a, b, c, d, a^-, b^-, c^-, d^-$ be distinct letters and let A_0, B_0, C, D_0 be linear sequences. Then, surface sets Q_j^k are defined as follows for $j = 1, 2, 3, \cdots, 24$:

$Q_1^k = \{A_kB_kCD_k\}$	$Q_2^k = \{A_kCD_kaB_ka^-\}$	$Q_3^k = \{A_kB_kCaD_ka^-\}$
$Q_4^k = \{A_kB_kaCD_ka^-\}$	$Q_5^k = \{A_kD_kaB_kCa^-\}$	$Q_6^k = \{A_kD_kCB_k\}$
$Q_7^k = \{B_kCD_kaA_ka^-\}$	$Q_8^k = \{B_kD_kCaA_ka^-\}$	$Q_9^k = \{A_kB_kD_kC\}$
$Q_{10}^k = \{A_kD_kCaB_ka^-\}$	$Q_{11}^k = \{A_kB_kD_kaCa^-\}$	$Q_{12}^k = \{A_kD_kB_kaCa^-\}$
$Q_{13}^k = \{A_kCB_kD_k\}$	$Q_{14}^k = \{A_kCB_kaD_ka^-\}$	$Q_{15}^k = \{A_kCD_kB_k\}$
$Q_{16}^k = \{A_kCaB_kD_ka^-\}$	$Q_{17}^k = \{A_kD_kB_kC\}$	$Q_{18}^k = \{CD_kaA_ka^-bB_kb^-\}$
$Q_{19}^k = \{B_kD_kaA_ka^-bCb^-\}$	$Q_{20}^k = \{B_kCaA_ka^-bD_kb^-\}$	$Q_{21}^k = \{A_kD_kaB_ka^-bCb^-\}$
$Q_{22}^k = \{A_kCaB_ka^-bD_kb^-\}$	$Q_{23}^k = \{A_kB_kaCa^-bD_kb^-\}$	$Q_{24}^k = \{A_kaB_ka^-bCb^-cD_kc^-\}$

where $k = 0$ and 1, $A_1 \in \{dA_0, A_0d\}$, $(B_1, D_1) \in \{(B_0d^-, D_0), (B_0, d^-D_0)\}$ and $a, a^-, b, b^-, c, c^-, d, d^- \notin ABCD$. Let $f_{Q_j^0}(x)$ denote the genus polynomial of Q_j^0. If $A_1^0A_0^0D_0B_1B_2^0C_2^0C_1D_1 = \emptyset$, then $f_{Q_j^0}(x) = 1$. Otherwise, suppose that $f_{Q_j^0}(x)$ are given for $1 \leqslant j \leqslant 24$. Then,

Theorem 3.1 *Let $g_{i_j}(n)$ be the number of surfaces with genus i in Q_j^n. Each of the following holds.*

$$g_{i_j}(1)=\begin{cases} g_{i_2}(0)+g_{i_3}(0)+g_{i_4}(0)+g_{i_5}(0), & \text{if } j=1\\ g_{i_{21}}(0)+g_{i_{22}}(0)+g_{(i-1)_1}(0)+g_{(i-1)_{15}}(0), & \text{if } j=2\\ g_{i_{22}}(0)+g_{i_{23}}(0)+g_{(i-1)_1}(0)+g_{(i-1)_{17}}(0), & \text{if } j=3\\ g_{i_4}(0)+g_{i_{18}}(0)+g_{(i-1)_6}(0)+g_{(i-1)_9}(0), & \text{if } j=4\\ g_{i_5}(0)+g_{i_{20}}(0)+g_{(i-1)_6}(0)+g_{(i-1)_{13}}(0), & \text{if } j=5\\ 2g_{i_6}(0)+2g_{i_8}(0), & \text{if } j=6\\ 2g_{(i-1)_{15}}(0)+2g_{(i-1)_{17}}(0), & \text{if } j=7 \text{ and } 16\\ 4g_{(i-1)_6}(0), & \text{if } j=8\\ 2g_{i_4}(0)+2g_{i_{10}}(0), & \text{if } j=9\\ g_{i_{10}}(0)+g_{i_{18}}(0)+g_{(i-1)_6}(0)+g_{(i-1)_9}(0), & \text{if } j=10\\ 2g_{i_{21}}(0)+2g_{i_{23}}(0), & \text{if } j=11\\ 2g_{i_{12}}(0)+2g_{i_{19}}(0), & \text{if } j=12\\ 2g_{i_5}(0)+2g_{i_{14}}(0), & \text{if } j=13\\ g_{i_{14}}(0)+g_{i_{20}}(0)+g_{(i-1)_6}(0)+g_{(i-1)_{13}}(0), & \text{if } j=14\\ g_{i_7}(0)+g_{i_{12}}(0)+g_{i_{15}}(0)+g_{i_{16}}(0), & \text{if } j=15\\ g_{i_7}(0)+g_{i_{12}}(0)+g_{i_{16}}(0)+g_{i_{17}}(0), & \text{if } j=17\\ 2g_{(i-1)_4}(0)+2g_{(i-1)_{10}}(0), & \text{if } j=18\\ 4g_{(i-1)_{12}}(0), & \text{if } j=19\\ 2g_{(i-1)_5}(0)+2g_{(i-1)_{14}}(0), & \text{if } j=20\\ g_{i_{21}}(0)+g_{i_{24}}(0)+g_{(i-1)_{11}}(0)+g_{(i-1)_{12}}(0), & \text{if } j=21\\ g_{(i-1)_2}(0)+g_{(i-1)_3}(0)+g_{(i-1)_{10}}(0)+g_{(i-1)_{14}}(0), & \text{if } j=22\\ g_{i_{23}}(0)+g_{i_{24}}(0)+g_{(i-1)_{11}}(0)+g_{(i-1)_{12}}(0), & \text{if } j=23\\ 2g_{(i-1)_{21}}(0)+2g_{(i-1)_{23}}(0), & \text{if } j=24 \end{cases}$$

Proof We shall prove the equation for $g_{i_6}(1)$, and the proofs for others are similar. Let

$U_1=\{A_0dd^-D_0CB_0\}$ $\qquad U_2=\{dA_0D_0CB_0d^-\}$

$U_3=\{A_0dD_0CB_0d^-\}$ $\qquad U_4=\{dA_0d^-D_0CB_0\}$.

By the definition of Q_6^1, we have $Q_6^1=\{U_1,U_2,U_3,U_4\}$. By the definition of g_i,

$$g_{i_6}(1)=g_i(U_1)+g_i(U_2)+g_i(U_3)+g_i(U_4).$$

By Op3,

$$A_0dd^-D_0CB_0\sim A_0D_0CB_0, \text{ and } dA_0D_0CB_0d^-=A_0D_0CB_0d^-d\sim A_0D_0CB_0.$$

It follows that

$$g_i(U_1)=g_i(U_2)=g_{i_6}(0). \tag{8}$$

By Lemma 2.3 (i) and Op2, we have

$$A_0dD_0CB_0d^-=D_0CB_0d^-A_0d\sim B_0D_0Cd^-A_0d\sim B_0D_0CaA_0a^-$$

and

$$dA_0d^-D_0CB_0 = B_0D_0CdA_0d^- \sim B_0D_0CaA_0a^-.$$

So

$$g_i(U_3) = g_i(U_4) = g_{i_8}(0). \tag{9}$$

Combining (1) and (2), we have

$$g_{i_6}(1) = 2g_{i_6}(0) + 2g_{i_8}(0).$$

§4. Embedding Surfaces of a Cubic Graph

Given a cubic graph G with n non-tree edges y_l $(1 \leqslant l \leqslant n)$, suppose that T is a spanning tree such that T contains the longest path of G and that $\widetilde{T}$ is an associated joint tree. Let X_l, Y_l, Z_l and F_l be linear sequences for $1 \leq l \leq n$ such that $X_l \cup Y_l = y_l$, $Z_l \cup F_l = y_l^-$, $X_l \neq Y_l$ and $Z_l \neq F_l$.

RECORD RULE: Choose a vertex u incident with two semi-edges as a starting vertex and travel $\widetilde{T}$ along with tree edges of $\widetilde{T}$. In order to write down surfaces, we shall consider three cases below.

Case 1: If v is incident with two semi-edges y_s and y_t. Suppose that the linear sequence is R when one arrives v. Then, write down $RX_sy_tY_s$ going away from v.

Case 2: If v is incident with one semi-edge y_s. Suppose that R_1 is the linear sequence when one arrives v in the first time. Then the sequence is R_1X_s when one leaves v in the first time. Suppose that R_2 is the linear sequence when one arrives v in the second time. Then the sequence is R_2Y_s when one leaves v in the second time.

Case 3: If v is not incident with any semi-edge. Suppose that R_1, R_2 and R_3 are, respectively, the linear sequences when one leaves v in the first time, the second time and the third time. Then, the sequences are $(R_2/R_1)R_1(R_3/R_2)$ and R_3 when one leaves v in the third time.

Here, $1 \leq s, t \leq n$ and $s \neq t$. If v is incident with a semi-edge y_s^-, then replace X_s with Z_s and replace Y_s with F_s.

Lemma 4.1 *There is a bijection between embedding surfaces of a cubic graph and surfaces obtained by the record rule.*

Proof Let T be a spanning tree such that $\widetilde{T}$ is a joint tree of G above. Suppose that σ_v is

a rotation of v and that R_1, R_2 and R_3 are given above.

$$\sigma_v = \begin{cases} (y_s, y_t, e_r), & \text{if } X_s = y_s \text{ or } F_s = y_s^- \\ & \quad \text{and } v \text{ is incident with } y_s, y_t \text{ and } e_r; \\ (y_t, y_s, e_r), & \text{if } Y_s = y_s \text{ or } Z_s = y_s^- \\ & \quad \text{and } v \text{ is incident with } y_s, y_t \text{ and } e_r; \\ (y_s, e_1, e_2), & \text{if } X_s = y_s \text{ or } F_s = y_s^- \\ & \quad \text{and } v \text{ is incident with } y_s, e_p \text{ and } e_q; \\ (e_1, y_s, e_2), & \text{if } Y_s = y_s \text{ or } Z_s = y_s^- \\ & \quad \text{and } v \text{ is incident with } y_s, e_p \text{ and } e_q; \\ (e_1, e_2, e_3), & \text{if the linear sequence is } R_3 \\ & \quad \text{and } v \text{ is incident with } e_p, e_q \text{ and } e_r; \\ (e_2, e_1, e_3), & \text{if the linear sequence is } (R_2/R_1)R_1(R_3/R_2) \\ & \quad \text{and } v \text{ is incident with } e_p, e_q \text{ and } e_r \end{cases}$$

where e_p, e_q and e_r are tree-edges for $1 \leqslant p, q, r \leqslant 2n-3$ and $e_p \neq e_q \neq e_r$ for $p \neq q \neq r$. Hence the conclusion holds. □

By the definitions for X_l, Y_l, Z_l and F_l, we have the following observation:

Observation 4.2 A surface set $H^{(0)}$ of G has properties below.
(1) Either X_l, $Y_l \in H^{(0)}$ or X_l, $Y_l \notin H^{(0)}$;
(2) Either Z_l, $F_l \in H^{(0)}$ or Z_l, $F_l \notin H^{(0)}$;
(3) If for some l with $1 \leqslant l \leqslant n$, $X_l, Y_l, Z_l, F_l \in H^{(0)}$, then $H^{(0)}$ has one of the following forms $X_lA^{(0)}Y_lB^{(0)}Z_lC^{(0)}F_l\,D^{(0)}$, $Y_lA^{(0)}X_lB^{(0)}Z_lC^{(0)}F_lD^{(0)}$, $X_lA^{(0)}Y_lB^{(0)}F_lC^{(0)}Z_lD^{(0)}$ or $Y_lA^{(0)}X_lB^{(0)}F_lC^{(0)}Z_lD^{(0)}$. These forms are regarded to have no difference through this paper.

If either $X_l \in H^{(0)}, Z_l \notin H^{(0)}$ or $X_l \notin H^{(0)}, Z_l \in H^{(0)}$, then replace X_l, Y_l, Z_l and F_l according to the definition of X_l, Y_l, Z_l and F_l.

RECURSION RULE: Given a surface set $H^{(0)} = \{X_lA^{(0)}Y_lB^{(0)}Z_lC^{(0)}F_lD^{(0)}\}$ where $A^{(0)}, B^{(0)}$, $C^{(0)}$ and $D^{(0)}$ are linear sequences.

Step 1. Let $A_0 = A^{(0)}$, $B_0 = B^{(0)}$, $C = C^{(0)}$ and $D_0 = D^{(0)}$. Q_j^1 is obtained for $2 \leqslant j \leqslant 5$. Then $H_j^{(1)}$ is obtained by replacing a, a^- and Q_j^1 with a_1, a_1^- and $H_j^{(1)}$ respectively.

Step 2. Given a surface set $H^{(k)}_{j_1,j_2,j_3,\cdots,j_k}$ for a positive integer k and $2 \leqslant j_1, j_2, j_3, \cdots, j_k \leqslant 5$, without loss of generality, suppose that $H^{(k)}_{j_1,j_2,j_3,\cdots,j_k} = \{X_sA^{(k)}Y_sB^{(k)}Z_sC^{(k)}F_sD^{(k)}\}$ where $A^{(k)}, B^{(k)}, C^{(k)}$ and $D^{(k)}$ are linear sequences for certain s $(1 \leqslant s \leqslant n)$. Let $A_0 = A^{(k)}$, $B_0 = B^{(k)}$, $C = C^{(k)}$ and $D_0 = D^{(k)}$. Q_j^1 is obtained for $2 \leqslant j \leqslant 5$. Then $H^{(k+1)}_{j_1,j_2,j_3,\cdots,j_k,j}$ is obtained by replacing a, a^- and Q_j^1 with a_{k+1}, a_{k+1}^- and $H^{(k+1)}_{j_1,j_2,j_3,\cdots,j_k,j}$ respectively.

Some surface sets $H^{(m)}_{j_1,j_2,j_3,\cdots,j_m}$ which contain a_l, a_l^-, y_l, y_l^- can be obtained by using step 2 for a positive integer m, $2 \leqslant j_1, j_2, j_3, \cdots, j_m \leqslant 5$ and $1 \leqslant l \leqslant n$. It is easy to compute $f_{H^{(m)}_{j_1,j_2,j_3,\cdots,j_m}}(x)$.

By Theorem 3.7,

$$\begin{aligned} g_i(H^{(r)}_{j_1,j_2,j_3,\cdots,j_r}) &= g_i(H^{(r+1)}_{j_1,j_2,j_3,\cdots,j_r,2}) + g_i(H^{(r+1)}_{j_1,j_2,j_3,\cdots,j_r,3}) \\ &+ g_i(H^{(r+1)}_{j_1,j_2,j_3,\cdots,j_r,4}) + g_i(H^{(r+1)}_{j_1,j_2,j_3,\cdots,j_r,5}), \qquad (1) \\ &\text{if } 0 \le r \le m-1, 2 \le j_1, j_2, j_3, \cdots, j_r \le 5. \end{aligned}$$

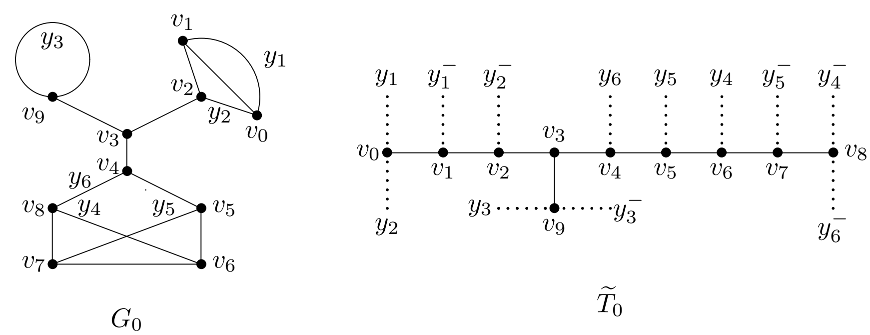

Fig.1: G_0 and $\widetilde{T}_0$

Example 4.3 The graph G_0 is given in Fig.1. A joint tree $\widetilde{T}_0$ is obtained by splitting non-tree edges y_l $(1 \leqslant l \leqslant 6)$. Travel $\widetilde{T}_0$ by regarded v_0 as a starting point. By using record rule we obtain surface sets

$$\{X_1 y_2 Y_1 Z_1 Z_2 Z_3 y_3 F_3 Y_6 Y_5 Y_4 Z_5 Z_4 y_6^- F_4 F_5 X_4 X_5 X_6 F_2 F_1\}$$

and

$$\{X_1 y_2 Y_1 Z_1 Z_2 Y_6 Y_5 Y_4 Z_5 Z_4 y_6^- F_4 F_5 X_4 X_5 X_6 F_2 F_1 Z_3 y_3 F_3\}.$$

By replacing Z_2, F_2, Z_3, F_3, X_6 and Y_6 according the definition 16 surface sets U_r $(1 \leqslant r \leqslant 16)$ are listed below.

$$U_1 = \{X_1 y_2 Y_1 Z_1 y_2^- y_3^- y_3 y_6 Y_5 Y_4 Z_5 Z_4 y_6^- F_4 F_5 X_4 X_5 F_1\}$$

$$U_2 = \{X_1 y_2 Y_1 Z_1 y_2^- y_3^- y_3 Y_5 Y_4 Z_5 Z_4 y_6^- F_4 F_5 X_4 X_5 y_6 F_1\}$$

$$U_3 = \{X_1 y_2 Y_1 Z_1 y_2^- y_3 y_3^- y_6 Y_5 Y_4 Z_5 Z_4 y_6^- F_4 F_5 X_4 X_5 F_1\}$$

$$U_4 = \{X_1 y_2 Y_1 Z_1 y_2^- y_3 y_3^- Y_5 Y_4 Z_5 Z_4 y_6^- F_4 F_5 X_4 X_5 y_6 F_1\}$$

$$U_5 = \{X_1 y_2 Y_1 Z_1 y_3^- y_3 y_6 Y_5 Y_4 Z_5 Z_4 y_6^- F_4 F_5 X_4 X_5 y_2^- F_1\}$$

$$U_6 = \{X_1 y_2 Y_1 Z_1 y_3^- y_3 Y_5 Y_4 Z_5 Z_4 y_6^- F_4 F_5 X_4 X_5 y_6 y_2^- F_1\}$$

$$U_7 = \{X_1 y_2 Y_1 Z_1 y_3 y_3^- y_6 Y_5 Y_4 Z_5 Z_4 y_6^- F_4 F_5 X_4 X_5 y_2^- F_1\}$$

$$U_8 = \{X_1 y_2 Y_1 Z_1 y_3 y_3^- Y_5 Y_4 Z_5 Z_4 y_6^- F_4 F_5 X_4 X_5 y_6 y_2^- F_1\}$$

$$U_9 = \{X_1 y_2 Y_1 Z_1 y_2^- y_6 Y_5 Y_4 Z_5 Z_4 y_6^- F_4 F_5 X_4 X_5 F_1 y_3^- y_3\}$$

$$U_{10}=\{X_1y_2Y_1Z_1y_2^-Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_6F_1y_3^-y_3\}$$

$$U_{11}=\{X_1y_2Y_1Z_1y_2^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5F_1y_3y_3^-\}$$

$$U_{12}=\{X_1y_2Y_1Z_1y_2^-Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_6F_1y_3y_3^-\}$$

$$U_{13}=\{X_1y_2Y_1Z_1y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_2^-F_1y_3^-y_3\}$$

$$U_{14}=\{X_1y_2Y_1Z_1Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_6y_2^-F_1y_3^-y_3\}$$

$$U_{15}=\{X_1y_2Y_1Z_1y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_2^-F_1y_3y_3^-\}$$

$$U_{16}=\{X_1y_2Y_1Z_1Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5y_6y_2^-F_1y_3y_3^-\}.$$

The genus distribution of U_r can be obtained by using the recursion rule. Since the method is similar, we shall calculate the genus distribution of U_1 and leave the calculation of genus distribution for others to readers.

U_1 is reduced to $\{X_1y_2Y_1Z_1y_2^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5F_1\}$ by Op2. Let $H^{(0)}=S_1$, $A_0=y_2$, $C_0=y_2^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5$ and $B_0=D_0=\emptyset$. Then $H_2^{(1)}=H_3^{(1)}=\{y_2y_2^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5\}$ and $H_4^{(1)}=H_5^{(1)}=\{y_2a_1y_2^-y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5a_1^-\}$.

$H_2^{(1)}$ is reduced to $\{y_6Y_5Y_4Z_5Z_4y_6^-F_4F_5X_4X_5\}$ by Op2. Let $A_0=X_5y_6Y_5$, $B_0=Z_5$, $C_0=y_6^-$ and $D_0=F_5$. Then $H_{2,2}^{(2)}=\{X_5y_6Y_5y_6^-F_5a_2Z_5a_2^-\}$, $H_{2,3}^{(2)}=\{X_5y_6Y_5Z_5y_6^-a_2F_5a_2^-\}$, $H_{2,4}^{(2)}=\{X_5y_6Y_5Z_5a_2y_6^-F_5a_2^-\}$ and $H_{2,5}^{(2)}=\{X_5y_6Y_5F_5a_2Z_5y_6^-a_2^-\}$. $H_{4,2}^{(2)}=\{X_5a_1^-y_2a_1y_2^-y_6Y_5y_6^-F_5a_2Z_5a_2^-\}$, $H_{4,3}^{(2)}=\{X_5a_1^-y_2a_1y_2^-y_6Y_5Z_5y_6^-a_2F_5a_2^-\}$, $H_{4,4}^{(2)}=\{X_5a_1^-y_2a_1y_2^-y_6Y_5Z_5a_2y_6^-F_5a_2^-\}$ and $H_{4,5}^{(2)}=\{X_5a_1^-y_2a_1y_2^-y_6Y_5F_5a_2Z_5y_6^-a_2^-\}$ by letting $A_0=X_5a_1^-y_2a_1y_2^-y_6Y_5$, $B_0=Z_5$, $C_0=y_6^-$ and $D_0=F_5$.

Similarly, $H_{2,2,2}^{(3)}=\{y_6a_2a_2^-a_3y_6^-a_3^-\}$, $H_{2,2,3}^{(3)}=\{y_6y_6^-a_2a_3a_2^-a_3^-\}$, $H_{2,2,4}^{(3)}=\{y_6y_6^-a_3a_2a_2^-a_3^-\}$ and $H_{2,2,5}^{(3)}=\{y_6a_2^-a_3y_6^-a_2a_3^-\}$. $H_{2,3,2}^{(3)}=\{y_6y_6^-a_2a_2^-\}$, $H_{2,3,3}^{(3)}=\{y_6y_6^-a_2a_3a_2^-a_3^-\}$, $H_{2,3,4}^{(3)}=\{y_6a_3y_6^-a_2a_2^-a_3^-\}$ and $H_{2,3,5}^{(3)}=\{y_6a_2^-a_3y_6^-a_2a_3^-\}$. $H_{2,4,2}^{(3)}=\{y_6a_2y_6^-a_2^-\}$, $H_{2,4,3}^{(3)}=\{y_6a_2y_6^-a_3a_2^-a_3^-\}$, $H_{2,4,4}^{(3)}=\{y_6a_3a_2y_6^-a_2^-a_3^-\}$ and $H_{2,4,5}^{(3)}=\{y_6a_2^-a_3a_2y_6^-a_3^-\}$. $H_{2,5,2}^{(3)}=\{y_6a_2y_6^-a_2^-\}$, $H_{2,5,3}^{(3)}=\{y_6a_2a_3y_6^-a_2^-a_3^-\}$, $H_{2,5,4}^{(3)}=\{y_6a_3a_2y_6^-a_2^-a_3^-\}$ and $H_{2,5,5}^{(3)}=\{y_6y_6^-a_2^-a_3a_2a_3^-\}$. $H_{4,2,2}^{(3)}=\{a_1^-y_2a_1y_2^-y_6a_2a_2^-a_3y_6^-a_3^-\}$, $H_{4,2,3}^{(3)}=\{a_1^-y_2a_1y_2^-y_6y_6^-a_2a_3a_2^-a_3^-\}$, $H_{4,2,4}^{(3)}=\{a_1^-y_2a_1y_2^-y_6y_6^-a_3a_2a_2^-a_3^-\}$ and $H_{4,2,5}^{(3)}=\{a_1^-y_2a_1y_2^-y_6a_2^-a_3y_6^-a_2a_3^-\}$. $H_{4,3,2}^{(3)}=\{a_1^-y_2a_1y_2^-y_6y_6^-a_2a_2^-\}$, $H_{4,3,3}^{(3)}=\{a_1^-y_2a_1y_2^-y_6y_6^-a_2a_3a_2^-a_3^-\}$, $H_{4,3,4}^{(3)}=\{a_1^-y_2a_1y_2^-y_6a_3y_6^-a_2a_2^-a_3^-\}$ and $H_{4,3,5}^{(3)}=\{a_1^-y_2a_1y_2^-y_6a_2^-a_3y_6^-a_2a_3^-\}$. $H_{4,4,2}^{(3)}=\{a_1^-y_2a_1y_2^-y_6a_2y_6^-a_2^-\}$, $H_{4,4,3}^{(3)}=\{a_1^-y_2a_1y_2^-y_6a_2y_6^-a_3a_2^-a_3^-\}$, $H_{4,4,4}^{(3)}=\{a_1^-y_2a_1y_2^-y_6a_3a_2y_6^-a_2^-a_3^-\}$ and $H_{4,4,5}^{(3)}=\{a_1^-y_2a_1y_2^-y_6a_2^-a_3a_2y_6^-a_3^-\}$. $H_{4,5,2}^{(3)}=\{a_1^-y_2a_1y_2^-y_6a_2y_6^-a_2^-\}$, $H_{4,5,3}^{(3)}=\{a_1^-y_2a_1y_2^-y_6a_2a_3y_6^-a_2^-a_3^-\}$, $H_{4,5,4}^{(3)}=\{a_1^-y_2a_1y_2^-y_6a_3a_2y_6^-a_2^-a_3^-\}$ and $H_{4,5,5}^{(3)}=\{a_1^-y_2a_1y_2^-y_6y_6^-a_2^-a_3a_2a_3^-\}$.

By using (1),

$$f_{U_1}(x)=4+32x+28x^2.$$

Thus,

$$f_{G_0}(x)=64+512x+448x^2.$$

§5. Genus Distribution for a Graph

Theorem 5.1 *Given a graph, the genus distribution of G is determined by using the genus distribution of some cubic graphs.*

Proof Given a finite graph G_0, suppose that u is adjacent to $k+1$ distinct vertices v_0, v_1, $v_2, \cdots, v_k$ of G_0 with $k \geq 3$. Actually, the supposition always holds by subdividing some edges of G.

A *distribution decomposition* of a graph is defined below: add a vertex u_s of valence 3 such that u_s is adjacent to u, v_0 and v_s for each s with $1 \leq s \leq k$ and then obtain a graph G_s by deleting the edges uv_0 and uv_s.

Choose the spanning trees T_s of G_s such that uv_s, uu_s and u_sv_s are tree edges for $0 \leq s \leq k$. Consider a joint tree $\widetilde{T}_0$ of G. Let $\widetilde{T}_s^*$ be the maximal joint tree of $\widetilde{T}_0$ such that $v_s \in V(T_s^*)$ and $v_t \notin V(T_s^*)$ for $t \neq s$ and $0 \leqslant s, t \leqslant k$.

Let v_s be the starting vertex of $\widetilde{T}_s^*$ for $0 \leqslant s \leqslant k$. Suppose that $\mathcal{A}_s$ is the set of all sequences by travelling $\widetilde{T}_s^*$ and that Q_s is the embedding surface set of G_s. Then

$$Q_0 = \{A_0 A_{r_1} A_{r_2} A_{r_3} \cdots A_{r_k} | A_{r_p} \in \mathcal{A}_{r_p},\ 1 \leqslant r_p \leqslant k, r_p \neq r_q \text{ for } p \neq q\}$$

and for $1 \leqslant s \leqslant k$

$$Q_s = \{A_0 A_s A_{r_1} A_{r_2} A_{r_3} \cdots A_{r_{k-1}}, A_0 A_{r_1} A_{r_2} A_{r_3} \cdots A_{r_{k-1}} A_s | A_{r_p} \in \mathcal{A}_{r_p},$$
$$1 \leqslant r_p \leqslant k, r_p \neq s, 1 \leqslant p, q \leqslant k-1, \text{ and } r_p \neq r_q \text{ for } p \neq q\}.$$

Let $f_{Q_s}(x)$ denote the genus distribution of Q_s. It is obvious that

$$f_{Q_0}(x) = \frac{1}{2} \sum_{s=1}^{k} f_{Q_s}(x).$$

Thus,

$$f_{G_0}(x) = \frac{1}{2} \sum_{s=1}^{k} f_{G_s}(x).$$

Since G_0 has finite vertices, the genus distribution of G_0 can be transformed into those of some cubic graphs in homeomorphism by using the distribution decomposition. □

Next we give a simple application of Theorem 5.1.

Example 5.2 The graph W_4 is shown in Fig.2. In order to calculate its genus distribution, we use the distribution decomposition and then we obtain three graph G_s for $1 \leqslant s \leqslant 3$ (Fig.2). It is obvious that G_2 are isomorphic to Möbius ladder ML_3 and G_s are isomorphic to Ringel ladder RL_2 for $s = 1$ and 3. Since (see [8], [15])

$$f_{ML_3}(x) = 40x + 24x^2$$

and since (see [9], [15])

$$f_{RL_2}(x) = 2 + 38x + 24x^2,$$

$$
\begin{aligned}
f_{W_4}(x) &= \frac{1}{2}\sum_{s=1}^{3} f_{G_s}(x) \\
&= \frac{1}{2}[40x + 24x^2 + 2(2 + 38x + 24x^2)] \\
&= 2 + 58x + 36x^2.
\end{aligned}
$$

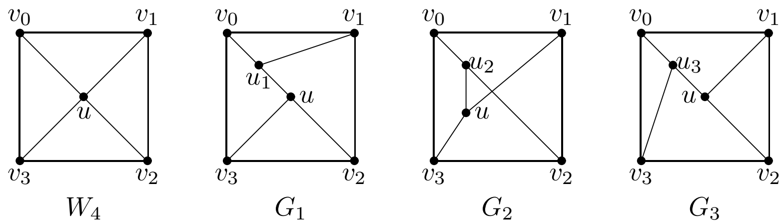

Fig.2: W_4 and G_s

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The greatest lesson in life is to know that even fools are right sometimes.

By Winston Churchill, a British statesman.

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[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.

Research papers

[8]K. K. Azad and Gunjan Agrawal, On the projective cover of an orbit space, *J. Austral. Math. Soc.* 46 (1989), 308-312.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

Figures: Figures should be drawn by TEXCAD in text directly, or as EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

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Mathematical Combinatorics, January 2010

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An International Book Series on Mathematical Combinatorics

ISBN 9781599731155